Problem-Based Mathematics II

1. (I.87.8) Given four numbers *a*, *b*, *c*, and *d*, one can ask for the distance from (a, b) to (c, d). Write a procedure for computing this distance, using the four numbers.

2. (II.23.10) Give an example of an equiangular polygon that is not equilateral.

3. (II.22.4) Triangle *ABC* is isosceles, with $AB = BC$, and angle *BAC* is 56 degrees. Find the remaining two angles of this triangle.

4. (II.22.5) Triangle *ABC* is isosceles, with $AB = BC$, and angle *ABC* is 56 degrees. Find the remaining two angles of this triangle.

5. (I.88.5) Both legs of a right triangle are 8 cm long. In simplest radical form, how long is the hypotenuse? How long would the hypotenuse be if both legs were *k* cm long?

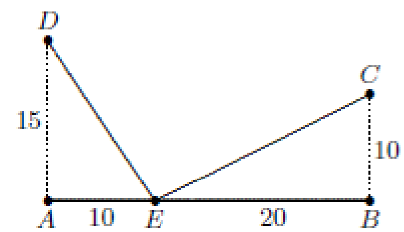
6. (I.88.6) The hypotenuse of a right triangle is twice as long as the shortest side, whose length is *m*. In terms of *m*, what is the length of the intermediate side?

7. (II.36.12) A *regular*, *n*-sided polygon has 18-degree exterior angles. Find the integer *n*.

8. (I.3.6) The area of the surface of a sphere is described by the formula $S = 4\pi r^2$, where *r* is the radius of the sphere. The Earth has a radius of 3960 miles and dry land forms approximately 29.2% of the Earth's surface. What is the area of the dry land on Earth? What is the surface area of the Earth's water?

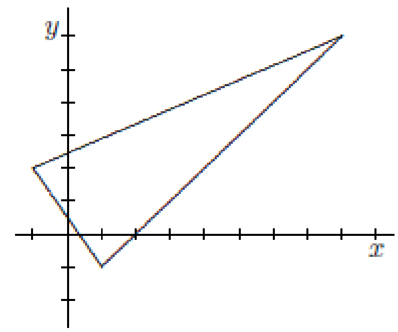
9. (II.15.4) Given $A = (-1, 5)$, $B = (x, 2)$, and $C = (4, -6)$, find the value of *x* that makes the path from *A* to *C* through *B* as short as possible.

10. (II.1.8) In the diagram, *AEB* is straight and angles *A* and *B* are right. Calculate the total distance $DE + EC$.



11. (II.1.9) (Continuation) If $AE = 15$ and $EB = 15$ instead, would $DE + EC$ be the same?

12. (II.1.10) (Continuation) You have seen that the length of $DE + EC$ depends on the value chosen for *AE*. Another way to say this is that *DE + EC is a function of AE*.
(a) Letting *x* stand for *AE* (and $30 - x$ for *EB*), write a formula for this function.
(b) Enter this formula into your calculator, graph it, and find the value of *x* that produces the *shortest* path from *D* to *C* through *E*.
(c) Using this value of x, draw an accurate picture of the path from *D* to *C* through *E*.
(d) Make a conjecture about *angles AED* and *BEC*. Use your protractor to test your conjecture.



1. (II.4.7) The sides of the triangle at right are formed by the graphs of $3x+2y=1$, $y=x-2$, and $-4x+9y=22$. Is the triangle isosceles? How do you know?

2. (II.8.4) The perimeter of an isosceles right triangle is 24 cm. How long are its sides?

3. (II.15.9) Let A = (0, 0), B = (4, 2), and C = (1, 3), find the size of angle CAB. Justify your answer.

4. (SAT problem) If the length of the largest side of a right triangle is 10 cm and one of the angles is 60°, what is the length of the smallest side?

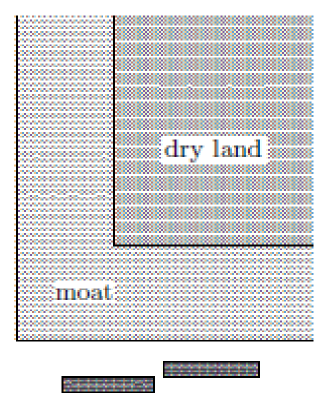
5. (II.41.13) The diagonals of a square have length 10. How long are the sides of the square?

6. (II.29.14) What do you call an (a) *equiangular quadrilateral*? (b) *equilateral quadrilateral*?

7. (II.40.7) If *MNPQRSTUV* is a regular polygon, then how large is each of its interior angles? If *MN* and *QP* are extended to meet at *A*, then how large is angle *PAN*?

8. (II.14.14) Let $A = (-6, -4)$, $B = (1, -1)$, $C = (0, -4)$, $D = (-7, -7)$. Show that the opposite sides of quadrilateral *ABCD* are parallel. Such a quadrilateral is called a *parallelogram*.

9. (II.14.4) When two lines intersect, four angles are formed. It is not hard to believe that the nonadjacent angles in this arrangement are *congruent*. If you had to prove this to a skeptic, what reasons would you offer?



10. (II.23.7) A castle is surrounded by a rectangular moat, which is of uniform width 12 feet. A corner is shown in the top view at right. The problem is to get across the moat to the dry land on the other side, without being able to use the drawbridge. All you have to work with are two rectangular planks, whose lengths are 11 feet and 11 feet, 9 inches. Find a way to get across.

11. (II.41.5) The altitudes of an equilateral triangle all have length 12 cm. How long are its sides?

12. (II.36.13) Let $A = (0, 0)$, $B = (7, 2)$, $C = (3, 4)$, $D = (3, 7)$, and $E = (-1, 5)$. Cameron walks the polygonal path *ABCDEA*, writing down the number of degrees turned at each corner. What is the sum of these five numbers? Notice that *ABCDE* is *not* a *convex* pentagon.

Problem-Based Mathematics II

1. (II.7.9) Draw the following segments:
(a) from (3,−1) to (10, 3); (b) from (1.3, 0.8) to (8.3, 4.8); (c) from $\left(\pi, \sqrt{2}\right)$ to $\left(7+\pi, 4+\sqrt{2}\right)$.
What do these segments have in common?

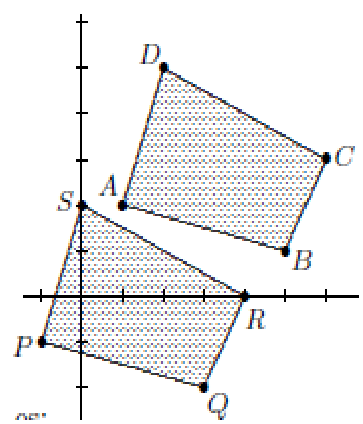
2. (II.7.10) (Continuation) The *directed segments* have the *same* length and the *same* direction. Each represents the *vector* [7, 4]. The *components* of the vector are the numbers 7 and 4.
(a) Find another example of a directed segment that represents this vector. The initial point of your segment is called the *tail* of the vector, and the final point is called the *head*.
(b) Which of the following directed segments represents [7, 4]? from (−2,−3) to (5,−1); from (−3,−2) to (11, 6); from (10, 5) to (3, 1); from (−7,−4) to (0, 0).

3. (II.11.1) Instead of saying that Cary moves *3 units left and 2 units up*, you can say that Cary's position is *displaced* by the vector [−3, 2]. Identify the following displacement vectors:
(a) Stacey starts at (2, 3) at 1 pm, and has moved to (5, 9) by 6 am;
(b) at noon, Eugene is at (3, 4); two hours earlier Eugene was at (6, 2);
(c) during a single hour, a small airplane flew 40 miles north and 100 miles west.

4. (II.8.8) A triangle has vertices A = (1, 2), B = (3, −5), and C = (6, 1). Triangle $A'B'C'$ is obtained by *sliding* triangle ABC 5 units to the right (in the positive x-direction, in other words) and 3 units up (in the positive y-direction). It is also customary to say that vector [5, 3] has been used to *translate* triangle ABC. What are the coordinates of A', B', and C'? By the way, "A prime" is the usual way of reading A'.

5. (II.8.9) (Continuation) When vector [h, k] is used to translate triangle ABC, it is found that the *image* of vertex A is (−3, 7). What are the images of vertices B and C?

6. (II.9.3) Let A = (1, 2), B = (5, 1), C = (6, 3), and D = (2, 5). Let P = (−1, −1), Q = (3, −2), R = (4, 0), and S = (0, 2). Use a vector to describe how quadrilateral $ABCD$ is related to quadrilateral $PQRS$.



7. (II.9.4) Let K = (3, 8), L = (7, 5), and M = (4, 1). Find coordinates for the vertices of the triangle that is obtained by using the vector [2, −5] to slide triangle KLM. How *far* does each vertex slide?

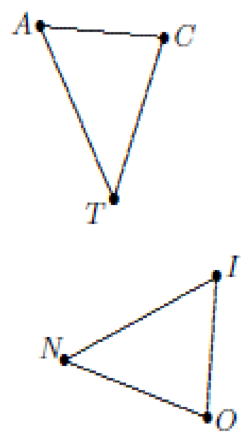
8. (II.16.3) If M is the midpoint of segment AB, how are vectors $\overrightarrow{AM}, \overrightarrow{AB}, \overrightarrow{MB},$ and $\overrightarrow{BM}$ related?

9. (II.49.1) The sides of a polygon are cyclically extended to form *rays*, creating one exterior angle at each vertex. Viewed from a great distance, what theorem does this figure illustrate?

Problem-Based Mathematics II

1. (II.11.5) Plot points $K = (0, 0)$, $L = (7, -1)$, $M = (9, 3)$, $P = (6, 7)$, $Q = (10, 5)$, and $R = (1, 2)$. Show that the triangles *KLM* and *RPQ* are congruent. Show also that neither triangle is a vector translation of the other. Describe how one triangle has been transformed into the other.

2. (II.11.6) (Continuation) If two figures are *congruent*, then their parts *correspond*. In other words, each part of one figure has been matched with a definite part of the other figure. In the triangle *PQR*, which angle corresponds to angle *M*? Which side corresponds to *KL*? In general, what can be said about corresponding parts of congruent figures? How might you confirm your hunch experimentally?



3. (II.16.6) Suppose that triangle *ACT* has been shown to be congruent to triangle *ION*, with vertices *A*, *C*, and *T* corresponding to vertices *I*, *O*, and *N*, respectively. It is customary to record this result by writing $\Delta ACT \cong \Delta ION$. Notice that corresponding vertices occupy corresponding positions in the equation. Let $B = (5, 2)$, $A = (-1, 3)$, $G = (-5, -2)$, $E = (1, -3)$, and $L = (0, 0)$. Using only these five labels, find as many pairs of congruent triangles as you can, and express the congruences accurately.

4. (II.16.7) (Continuation) How many ways are there of arranging the six letters of $\Delta ACT \cong \Delta ION$ to express the two-triangle congruence?

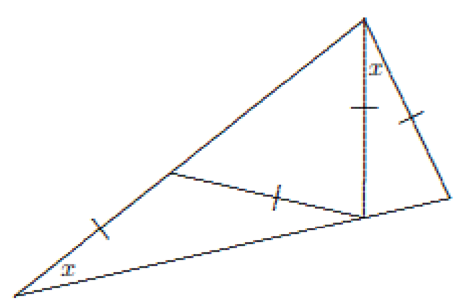
5. (II.16.8) What can be concluded about triangle *ABC* if it is given that
(a) $\Delta ABC \cong \Delta ACB$? (b) $\Delta ABC \cong \Delta BCA$?

6. (II.9.7) Let $A = (-5, 0)$, $B = (5, 0)$, and $C = (2, 6)$; let $K = (5, -2)$, $L = (13, 4)$, and $M = (7, 7)$. Verify that the length of each side of triangle *ABC* matches the length of a side of triangle *KLM*. Because of this data, it is natural to regard the triangles as being in some sense equivalent. It is customary to call the triangles *congruent*. The basis used for this judgment is called the *side-side-side* criterion. What can you say about the sizes of angles *ACB* and *KML*? What is your reasoning? What about the other angles?

7. (II.9.8) (Continuation) Are the triangles related by a vector translation? Why or why not?

8. (II.17.8) Let $A = (0, 0)$, $B = (1, 2)$, $C = (6, 2)$, $D = (2, -1)$, and $E = (1, -3)$. Show that angle *CAB* is the same size as angle *EAD*.

9. (II.49.9) In the figure at right, there are two *x*-degree angles, and four of the segments are congruent as marked. Find *x*.

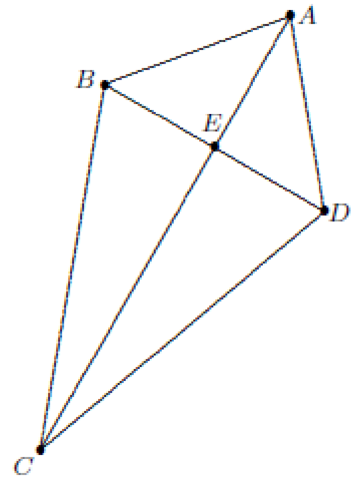


10. (II.11.3) A bug is initially at $(-3, 7)$. Where is the bug after being displaced by vector $[-7, 8]$?

Problem-Based Mathematics II

Here are two examples of proofs that do not use coordinates. Both proofs show how specific *given* information can be used to logically deduce *new* information. Each example concerns a *kite ABCD*, for which $AB = AD$ and $BC = DC$ is the given information. The first proof, which consists of simple text, shows that diagonal AC creates angles BAC and DAC of the same size.

Proof 1: Because AB = AD and BC = DC, and because the segment AC is shared by the triangles ABC and ADC, it follows from the SSS criterion that these triangles are congruent. Thus it is safe to assume that all the corresponding parts of these triangles are congruent as well (often abbreviated to *CPCTC*, as in proof 2 below.) In particular, angles BAC and DAC are the same size. ■



Now let E mark the intersection of diagonals AC and BD. The second proof, which is an example of a *two-column* proof, is written symbolically in outline form. It shows that the diagonals intersect perpendicularly. This proof builds on the first proof, which thus reappears as the first five lines.

Proof 2:

$AB = AD$	Given
$BC = DC$	Given
$AC = AC$	Shared side
$\Delta ABC \cong \Delta ADC$	SSS
$\angle BAC \cong \angle DAC$	CPCTC
E = intersection of AC and BD	
$AB = AD$	Given
$\angle BAE \cong \angle DAE$	Preceding CPCTC
$AE = AE$	Shared side
$\Delta ABE \cong \Delta ADE$	SAS
$\angle BEA \cong \angle DEA$	CPCTC
$\angle BEA$ and $\angle DEA$ supplementary	E is on BD
$\angle BEA$ is right	Definition of right angle

1. (II.18.1) In the fourth line, why would it have been wrong to write $\Delta ABC \cong \Delta ACD$?

2. (II.18.2) Refer to the kite data above and prove that angles ABC and ADC are also the same size.

3. (II.19.11) Refer to the data above and prove that one of the diagonals of a kite is bisected by the other.

1. (II.19.9) If a quadrilateral is equilateral, its diagonals are perpendicular. True or false? Prove your claim.

2. (II.19.10) The diagonals *AC* and *BD* of quadrilateral *ABCD* intersect at *O*. Given the information $AO = BO$ and $CO = DO$, what can you deduce about the lengths of the sides of the quadrilateral? Prove your response.

3. (II.23.8) Find k so that the vectors $[4,-3]$ and $[k,-6]$ (a) point in the same direction; (b) are perpendicular.

4. (II.21.10) Prove that one of the diagonals of a kite bisects two of the angles of the kite. What about the other diagonal—must it also be an *angle bisector*? Explain your response.

5. (II.41.7) Triangle *ABC* has $AB = AC$. The bisector of angle *B* meets *AC* at *D*. Extend side *BC* to *E* so that $CE = CD$. Triangle *BDE* should look isosceles. Is it? Explain.

6. (II.14.9) A line goes through the points (2, 5) and (6, −1). Let *P* be the point on this line that is closest to the origin. Calculate the coordinates of *P*.

7. (III.66.5) There are twenty-three red marbles and two blue marbles in a box. A marble is randomly chosen from the box, its color noted, then put back in the box. This process is repeated. What is the *probability* that **(a)** the first marble is blue? **(b)** the first four marbles are red? **(c)** the first four marbles are red and the fifth is blue?

8. (II.20.6) In quadrilateral *ABCD*, it is given that $AB = CD$ and $BC = DA$. Prove that angles *ACD* and *CAB* are the same size. N.B. If a polygon has more than three vertices, the *labeling convention* is to place the letters around the polygon in the order that they are listed. Thus *AC* should be one of the diagonals of *ABCD*.

9. (II.24.7) Find a vector that is perpendicular to the line $3x - 4y = 6$.

10. (II.21.8) Let $A = (1, 4)$, $B = (8, 0)$, and $C = (7, 8)$. Find the area of triangle *ABC*.

11. (II.11.10) Let $A = (3, 2)$ and $B = (7, -10)$. What is the displacement vector that moves point *A* onto point *B*? What vector moves *B* onto *A*?

12. (II.11.11) Explain why the segment from (a, b) to (c, d) has the same length as the segment from $(a+h, b+k)$ to $(c+h, d+k)$.

13. (II.21.12) If the diagonals of a quadrilateral bisect each other, then any two nonadjacent sides of the figure must have the same length. Prove that this is so.

Problem-Based Mathematics II

1. (IV.1.2) In many states, automobile license plates display six characters—three letters followed by a three-digit number, as in SSA-127. Would this system work adequately in Pennsylvania?

2. (II.17.5) Given $A = (6, 1)$, $B = (1, 3)$, and $C = (4, 3)$, find a *lattice point* P that makes $\overrightarrow{CP}$ perpendicular to $\overrightarrow{AB}$.

3. (II.17.6) (Continuation) Describe the set of points P for which $\overrightarrow{AB}$ and $\overrightarrow{CP}$ are perpendicular.

4. (II.24.6) Given the points $A = (0, 0)$, $B = (7, 1)$, and $D = (3, 4)$, find coordinates for the point C that makes quadrilateral $ABCD$ a parallelogram. What if the question had requested $ABDC$ instead?

5. (II.22.8) Find the area of the triangle whose vertices are $A = (-2, 3)$, $B = (6, 7)$, and $C = (0, 6)$.

6. (II.22.6) Show that two vectors $[a, b]$ and $[c, d]$ are perpendicular if, and only if, $ac + bd = 0$. The number $ac + bd$ is called the *dot product* of the vectors $[a, b]$ and $[c, d]$.

7. (II.11.9) Choose a point P on the line $2x + 3y = 7,$ and draw the vector $[2, 3]$ with its tail at P and its head at Q. Confirm that the vector is perpendicular to the line. What is the distance from Q to the line? Repeat the preceding, with a different choice for point P.

8. (II.12.2) Given the vector $[-5, 12]$, find the following vectors:
(a) same direction, twice as long
(b) same direction, length 1
(c) opposite direction, length 10
(d) opposite direction, length c

9. (II.12.3) *Some terminology*: When the components of the vector $[5, -7]$ are multiplied by a given number t, the result may be written either as $[5t, -7t]$ or as $t[5, -7]$. This is called the *scalar multiple* of vector $[5, -7]$ by the *scalar t*. Find components for the following scalar multiples:
(a) $[12, -3]$ by scalar 5
(b) $\left[\sqrt{5},\ \sqrt{10}\right]$ by scalar $\sqrt{5}$
(c) $\left[-\frac{3}{4},\ \frac{2}{3}\right]$ by scalar $-\frac{1}{2}+\frac{2}{6}$
(d) $\left[p,\ q\right]$ by scalar $\frac{p}{q}$

10. (II.12.4) Find the lengths of the following vectors:
(a) $[3, 4]$ (b) $2017[3, 4]$ (c) $\frac{2017}{5}[3, 4]$ (d) $t[3, 4]$ (e) $t[a, b]$

11. (II.24.8) Measurements are made on quadrilaterals $ABCD$ and $PQRS$, and it is found that angles A, B, and C are the same size as angles P, Q, and R, respectively, and that sides AB and BC are the same length as PQ and QR, respectively. Is this enough evidence to conclude that the quadrilaterals $ABCD$ and $PQRS$ are congruent? Explain.

Problem-Based Mathematics II

1. (II.31.10) Write an equation that says that vectors $[a,b]$ and $[m,n]$ are perpendicular.

2. (II.25.8) In triangle *ABC*, it is given that *CA* = *CB*. Points *P* and *Q* are marked on segments *CA* and *CB*, respectively, so that angles *CBP* and *CAQ* are the same size. Prove that *CP* = *CQ*.

3. (II.25.9) (Continuation) Segments *BP* and *AQ* intersect at *K*. Explain why you can be sure that quadrilateral *CPKQ* is a kite. You might want to consider triangles *AKP* and *BKQ*.

4. (II.13.10) After drawing the line $y=2x-1$ and marking the point *A* = (−2, 7), Kendall is trying to decide which point on the line is closest to *A*. The point *P* = (3, 5) looks promising. To check that *P* really is the point on $y=2x-1$ that is closest to *A*, what should Kendall do? *Is P* closest to *A*?

5. (II.16.9) Plot points *K* = (−4, −3), *L* = (−3, 4), *M* = (−6, 3), *X* = (0, −5), *Y* = (6, −3), and *Z* = (5, 0). Show that triangle *KLM* is congruent to triangle *XZY*. Describe a *transformation* that transforms *KLM* onto *XZY*. Where does this transformation send the point (−5, 0)?

6. (II.14.8) Find a point on the line $2x+y=8$ that is *equidistant* from the coordinate axes. How many such points are there?

7. (II.20.3) Let *A* = (−2, 3), *B* = (6, 7), and *C* = (−1, 6)
(a) Find an equation for the perpendicular bisector of *AB*.
(b) Find an equation for the perpendicular bisector of *BC*.
(c) Find coordinates for a point *K* that is equidistant from *A*, *B*, and *C*. By the way, if you draw a triangle connecting points A, B and C, then K is the *circumcenter* of that triangle.

8. (II.27.9) If the parts of two triangles are matched so that two angles of one triangle are congruent to the corresponding angles of the other, and so that a side of one triangle is congruent to the corresponding side of the other, then the triangles must be congruent. Justify this *angle-angle-corresponding side* (AAS) criterion for congruence. Would AAS be a valid test for congruence if the word *corresponding* were left out of the definition? Explain.

9. (II.15.8) Find a vector that translates the line $2x-3y=18$ onto the line $2x-3y=24$. (There is more than one correct answer.) Can you find the distance that separates these lines? You will have to decide what "distance" means in this context.

10. (II.25.6) Plot all points that are 3 units from the *x*-axis. Describe the configuration.

11. (II.25.7) Plot all points that are 3 units from the *x*-axis *and* 3 units from (5, 4). How many did you find?

12. (II.26.7) Simplify equation $\sqrt{(x-3)^2+(y-5)^2}=\sqrt{(x-7)^2+(y+1)^2}$. Interpret your result.

Problem-Based Mathematics II

1. (II.20.8) A *direction vector* for a line is any vector that joins two points on that line. Find a direction vector for $2x+5y=8.$ It is not certain that you and your classmates will get exactly the same answer. How should your answers be related, however?

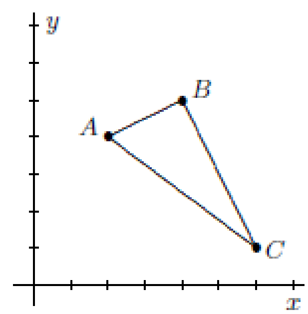
2. (II.20.9) (Continuation) Show that $[b, -a]$ is a direction vector for the line $ax + by = c$.

3. (II.20.10) (Continuation) Show that any direction vector for the line $ax+by=c$ must be perpendicular to $[a, b]$.

4. (I.50.2) Find values for a and b that make $ax+by=14$ parallel to $12-3y=4x.$ Is there more than one answer? If so, how are the different values for a and b related?

5. (II.23.9) The lines $3x + 4y = 12$ and $3x + 4y = 72$ are parallel. Explain why, then find the distance that separates these lines. You will have to decide what "distance" means in this context.

6. (II.9.9) Let $A = (2, 4)$, $B = (4, 5)$, and $C = (6, 1)$. Triangle ABC is shown at right. Draw two new triangles as follows:
(a) ΔPQR has $P = (11, 1)$, $Q = (10,-1)$, and $R = (6, 1)$;
(b) ΔKLM has $K = (8, 10)$, $L = (7, 8)$, and $M = (11, 6)$;

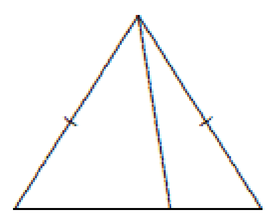


These triangles are not obtained from ABC by applying vector translations. Instead, each of the appropriate transformations is described by the suggestive names *reflection* or *rotation*. Decide which is which, with justification.

7. (II.28.12) Given: (5,2) is reflected onto (−1, 4) across a line. What is the equation of this line of reflection?

8. (II.25.10) A polygon that is both equilateral and equiangular is called *regular*. Prove that all diagonals of a regular *pentagon* (five sides) have the same length.

9. (II.26.12) Use the diagram to help you explain why SSA evidence is not by itself sufficient to justify the congruence of triangles. The tick marks designate segments that have the same length.



10. (II.17.9) Let $A = (-2, 4)$ and $B = (7, -2)$. Find the point Q on the line $y = 2$ that makes the total distance $AQ + BQ$ as small as possible.

11. (II.17.10) (Continuation) Let $A = (-2, 4)$ and $B = (7, 6)$. Find the point P on the line $y = 2$ that makes the total distance $AP + BP$ as small as possible.

12. (II.26.13) The diagonals of a kite are 6 cm and 12 cm long. Is it possible for the lengths of the sides of this kite to be in a 2-to-1 ratio?

Problem-Based Mathematics II

1. (II.28.3) Find the area of the triangle having sides 10, 10, and 5.

2. (II.27.12) A triangle that has a 13-inch side, a 14-inch side, and a 15-inch side has an area of 84 square inches. Accepting this fact, find the lengths of all three altitudes of this triangle.

3. (II.28.13) Draw a parallelogram whose adjacent edges are determined by vectors [2, 5] and [8, 0], placed so that they have a common initial point. This is called placing vectors *tail-to-tail*. Find the area of the parallelogram.

4. (II.28.13) (Continuation) Draw a parallelogram whose adjacent edges are determined by vectors [2, 5] and [7, −1], placed tail-to-tail. Find the area of the parallelogram.

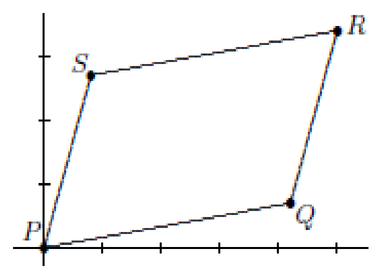
5. (II.26.5) Find coordinates for a point that is three times as far from the origin as (2, 3) is. Describe the configuration of all such points.

6. (II.47.9) Suppose that $ABCD$ is a parallelogram, in which $AB = 2BC$. Let M be the midpoint of segment AB. Prove that segments CM and DM bisect angles BCD and CDA, respectively. What is the size of angle CMD? Justify your response.

7. (II.25.11) Find coordinates for the point equidistant from (−1, 5), (8, 2), and (6, −2).

8. (II.29.9) The diagonals of quadrilateral $ABCD$ intersect perpendicularly at O. What can be said about quadrilateral $ABCD$?

9. (II.25.2) The figure at right shows a parallelogram $PQRS$, three of whose vertices are $P = (0, 0)$, $Q = (a, b)$, and $S = (c, d)$.
(a) Find the coordinates of R.
(b) Find the area of $PQRS$, and simplify your formula.



10. (II.25.5) Find points on the line $3x + 5y = 15$ that are equidistant from the coordinate axes.

11. (II.30.6) In quadrilateral $ABCD$, it is given that $\overrightarrow{AB} = \overrightarrow{DC}$. What kind of a quadrilateral is $ABCD$? What can be said about the vectors $\overrightarrow{AD}$ and $\overrightarrow{BC}$?

12. (II.29.6) Find the area of a triangle formed by placing the vectors [3, 6] and [7, 1] tail-to-tail.

13. (II.23.3) Given points $A = (0, 0)$ and $B = (-2, 7)$, find coordinates for C and D so that $ABCD$ is a square.

14. (II.33.5) Given parallelogram $PQRS$, let T be the intersection of the bisectors of angles P and Q. Without knowing the sizes of the angles of $PQRS$, calculate the size of angle PTQ.

1. (II.39.5) A *trapezoid* is a quadrilateral with exactly one pair of parallel sides. If the non-parallel sides have the same length, the trapezoid is *isosceles*. Make a diagram of an isosceles trapezoid whose sides have lengths 7 in, 10 in, 19 in, and 10 in. Find the *altitude* of this trapezoid (the distance that separates the parallel sides), then find the enclosed area

2. (III.88.4) How many different eight-letter words can be formed from the letters of *absolute*, using each letter once per word? The words do not have to actually spell anything, of course.

3. (III.88.4) (Continuation) How many different six-letter words can be formed from the letters of *vertex*, using each letter once per word? How about twelve-letter words formed from the letters of *mathematical*, using each letter once per word?

4. (II.34.11) Suppose that one of the *medians* of a triangle happens to be exactly half the length of the side to which it is drawn. What can be said about the angles of this triangle? Justify your response.

5. (II.34.12) (Continuation) Prove that the midpoint of the hypotenuse of a right triangle is equidistant from all three vertices of the triangle. What is this point called? How does this statement relate to the preceding?

6. (II.26.1) Let $E = (2, 7)$ and $F = (10, 1)$. On the line through E and F, there are two points that are 3 units from E. Find coordinates for both of them.

7. (II.37.1) Draw a triangle ABC, and let AM and BN be two of its medians, which intersect at G. Extend AM to the point P that makes $GM = MP$. Prove that $PBGC$ is a parallelogram.

8. (II.38.6) Suppose that triangle ABC has a right angle at B, that BF is the altitude drawn from B to AC, and that BN is the median drawn from B to AC. Find angles ANB and NBF, given that (a) angle C is 42 degrees; (b) angle C is 48 degrees.

9. (II.37.4) Triangle PQR has a right angle at P. Let M be the midpoint of QR, and let F be the point where the altitude through P meets QR. Given that angle FPM is 18 degrees, find the sizes of angles Q and R.

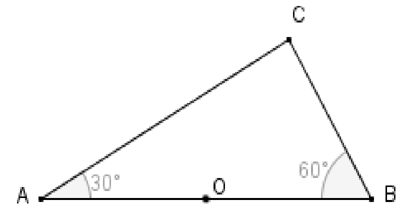
10. (II.27.3) You have recently seen that there is no generally reliable SSA criterion for triangle congruence. If the angle part of such a correspondence is a *right* angle, however, the criterion *is* reliable. Justify this so-called *hypotenuse-leg* criterion (which is abbreviated HL).

11. (II.39.12) A trapezoid has a 60-degree angle and a 45-degree angle. What are the other angles?

12. (II.39.13) A trapezoid has a 60-degree angle and a 120-degree angle. What are the other angles?

Problem-Based Mathematics II

1. (II.26.14) Translate the line $5x + 7y = 35$ by vector [3, 10]. Find an equation for the new line.



2. (II.63.5) As shown in the diagram at right, triangle *ABC* has a 30-degree angle at *A* and a 60-degree angle at *B*. Let *O* be the midpoint of *AB.*
(a) Draw the *circle* centered at *O* that goes through *A*.
(b) Explain why this circle also goes through *B* and *C*.
(c) Draw angle *BOC* and find its measure.

3. (Continuation) *BOC* is called a *central angle* of the circle because its vertex is at the center. Angle *BOC* and minor arc *BC* correspond to the same section of the circle, so we say that minor arc *BC* has the same *angular size* as angle *BOC*.
(a) What is the angular size of minor arc *BC*?
(b) What is the angular size of major arc *BAC*?
(c) What is the angular size of minor arc *AC*?
(d) What is the angular size of major arc *ABC*?
(e) How does the actual length of minor arc *AC* compare to the length of minor arc *BC*?

4. (From Dr. Sutula) A plane 6 cm from the center of a sphere intersects the sphere in a circle with diameter 16 cm. Find the radius and the volume of the sphere.

5. (II.57.11) Write an equation that describes all the points on the circle whose *center* is at the origin and whose *radius* is 13.

6. (II.40.2) Trapezoid *ABCD* has parallel sides *AB* and *CD*, a right angle at *D*, and the lengths $AB = 15$, $BC = 10$, and $CD = 7$. Find the length *DA*.

7. (II.40.6) What can be said about quadrilateral *ABCD*, if it has supplementary adjacent angles?

8. (II.28.11) When translation by vector [2, 5] is followed by translation by vector [5, 7], the net result can be achieved by applying a *single* translation; what is its vector?

9. (II.41.8) If *ABCD* is a quadrilateral, and *BD* bisects both angle *ABC* and angle *CDA*, then what sort of quadrilateral must *ABCD* be?

10. (II.40.10) Prove that an isosceles trapezoid must have two pairs of equal adjacent angles.

11. (II.40.11) (Continuation) The *converse* question: If a trapezoid has two pairs of equal adjacent angles, is it necessary that its non-parallel sides have the same length? Explain.

12. (II.39.11) The diagonals of a parallelogram always bisect each other. Is it possible for the diagonals of a trapezoid to bisect each other? Explain.

Problem-Based Mathematics II

1. (II.29.10) The *sum* of two vectors $[a,b]$ and $[p,q]$ is defined as $[a+p, b+q]$. Find the components of the vector $[2,3]+[-7,5]$. Graph $[2,3]$, $[-7,5]$, and their sum on a set of axes.

2. (II.41.6) It is given that the sides of an isosceles trapezoid have lengths 3 in, 15 in, 21 in, and 15 in. Make a diagram. Show that the diagonals intersect perpendicularly.

3. (II.29.13) Choose coordinates for three non-collinear points *A*, *B*, and *C*. Calculate components for the vectors $\overrightarrow{AB}, \overrightarrow{AC},$ and $\overrightarrow{AB}+\overrightarrow{AC},$ then translate point A by vector $\overrightarrow{AB}+\overrightarrow{AC}.$ Call the new point *D*. What kind of quadrilateral is *ABDC*?

4. (II.30.5) Simplify the sum of vectors $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}.$

5. (II.41.9) In quadrilateral *ABCD*, angles *ABC* and *CDA* are both bisected by *BD*, and angles *DAB* and *BCD* are both bisected by *AC*. What sort of quadrilateral must *ABCD* be?

6. (III.9.7) The *dot product* of vectors $\mathbf{u} = [a,b]$ and $\mathbf{v} = [m,n]$ is the number $\mathbf{u}\cdot\mathbf{v} = am+bn.$ In general, the dot product of two vectors is the sum of all the products of corresponding components. Let $\mathbf{u} = [-2,3]$, $\mathbf{v} = [0,4]$ and $\mathbf{w} = [-5,1].$ Calculate

(a) $\mathbf{u}\cdot\mathbf{v}$ (b) $\mathbf{v}\cdot\mathbf{w}$ (c) $\mathbf{w}\cdot\mathbf{v}$ (d) $\mathbf{w}\cdot\mathbf{w}$ (d) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})$ (e) $\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$

7. (III.11.10) Given a vector **u =** [a, b], the familiar *absolute-value* notation $|\mathbf{u}|$ is often used for its *magnitude*.

(a) Find $|\mathbf{u}|$ if u = [-3,4]

(b) Find $|\mathbf{u}|$ if u = [1,1]

(c) Find $|\mathbf{u}|$ if u = [a, b]

(d) Show that $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$. What theorem does this recall?

8. (II.46.3) Suppose that a quadrilateral is measured and found to have a pair of equal nonadjacent sides and a pair of equal nonadjacent angles. Is this enough evidence to conclude that the quadrilateral is a parallelogram? Explain.

9. (II.42.4) Is it possible for the diagonals of a parallelogram to have the same length? How about the diagonals of a trapezoid? How about the diagonals of a non-isosceles trapezoid?

10. (II.32.9) Let *ABCD* be a parallelogram. (a) Express $\overrightarrow{AC}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{BC}.$ (b) Express $\overrightarrow{AC}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AD}.$ (c) Express $\overrightarrow{BD}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AD}.$

11. (II.28.10) Let *P* = (2, 7), *B* = (6, 11), and *M* = (5, 2). Find a point *D* that makes $\overrightarrow{PB}=\overrightarrow{DM}$. What can you say about quadrilateral *PBMD*?

1. (II.64.8) Suppose that *MP* is a diameter of a circle centered at *O*, and *Q* is any other point on the circle. Draw the line through *O* that is parallel to *MQ*, and let *R* be the point where it meets minor arc *PQ*. Prove that *R* is the midpoint of minor arc *PQ*.

2. (II.49.3) Given a rectangular card that is 5 inches long and 3 inches wide, what does it mean for another rectangular card to have the *same shape*? Describe a couple of examples.

3. (II.58.3) Graph the circle whose equation is $x^2 + y^2 = 64$. What is its radius? What do the equations $x^2 + y^2 = 1$, $x^2 + y^2 = 40$, and $x^2 + y^2 = k$ all have in common? How do they differ?

4. (II.57.11) Write an equation that describes all the points on the circle whose *center* is at the origin and whose *radius* is *r*.

5. (II.30.7) Mark a lattice point on your graph paper. Define vector **u** (which can be handwritten $\vec{u}$) by moving 5 units to the right and 2 units up. Define vector **v** by moving 1 unit to the right and 3 units down. Diagram the vectors **u** + **v**, **u** − **v**, and 2**u** − 3**v**.

6. (II.30.8) Draw a parallelogram. Choose one of its vertices and let **u** and **v** be the vectors defined by the sides that originate at that vertex. Draw **u** + **v** and **u** − **v**. The vectors **u** and **v** represent the sides of the parallelogram; what do **u** + **v** and **u** − **v** represent?

7. (III.18.11) Pick two vectors **u** and **v**.
(a) Draw these vectors so that **v** originates at the head of **u.**
(b) Find **u + v** and draw it so that **u** and **u + v** have a common initial point
(c) Calculate $|\mathbf{u}|, |\mathbf{v}|$, and $|\mathbf{u}+\mathbf{v}|$, and verify that $|\mathbf{u}+\mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$.
(d) *The triangle inequality*. Explain why $|\mathbf{u}+\mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$ is true for any vectors **u** and **v.**

8. (II.41.3) A parallelogram has two 19-inch sides and two 23-inch sides. What is the range of possible lengths for the diagonals of this parallelogram?

9. (II.46.5) Let $P = (-15, 0)$, $Q = (5, 0)$, $R = (8, 21)$, and $S = (0, 15)$. Draw quadrilateral *PQRS* and measure its sides and angles. Pat and Kim are arguing about this shape. Pat claims that *PQRS* must be a parallelogram because it contains a pair of congruent opposite angles and a pair of congruent opposite sides. Kim says, sure, it has those properties but that does not necessarily mean that *PQRS* is a parallelogram. Settle their argument.

10. (II.36.7) *Midline Theorem*. Draw a triangle *ABC*, and let *M* and *N* be the midpoints of sides *AB* and *AC*, respectively. Express $\overrightarrow{BC}$ and $\overrightarrow{MN}$ in terms of $\boldsymbol{u} = \overrightarrow{AB}$ and $\boldsymbol{v} = \overrightarrow{AC}$.

11. (II.41.4) Is it possible for a trapezoid to have sides of lengths 3, 7, 5 and 11?

Problem-Based Mathematics II

1. (II.66.6) Triangle *ABC* has a 53-degree angle at *A*, and its circumcenter is at *K*.
(a) Draw a good picture of this triangle, and describe the process you used to find *K*. Then draw the circle centered at *K* that goes through *A*, *B*, and *C*.
(b) With your protractor, measure the size of angle *BKC*. What is the angular size of arc *BC*?
(c) Measure the angles *B* and *AKC* of your triangle. What is the angular size of arc *AC*?
(d) Using these measurements, find the angular size of *C*, *AKB* and arc *AB*.
(e) Angles *A*, *B* and *C* are *inscribed angles* because all three points are on the circle. Make a conjecture about arcs intercepted by inscribed angles.

2. (II.46.12) What is the radius of the smallest circle that encloses an equilateral triangle with 12-inch sides? What is the radius of the largest circle that will fit inside the same triangle?

3. (II.56.14) One stick measures 12 inches and another stick measures 24 inches. You break the longer stick at a random point. Now you have three sticks. What is the probability that they form a triangle?

4. (II.43.9) Draw an acute-angled triangle *ABC*, and mark points *P* and *Q* on sides *AB* and *AC*, respectively, so that $AB = 3AP$ and $AC = 3AQ$. Express $\overrightarrow{PQ}$ and $\overrightarrow{BC}$ in terms of $\mathbf{v} = \overrightarrow{AP}$ and $\mathbf{w} = \overrightarrow{AQ}$.

5. (II.46.10) Let *RICK* be a parallelogram, with *M* the midpoint of *RI*. Draw the line through *R* that is parallel to *MC*; it meets the extension of *IC* at *P*. Prove that $CP = KR$.

6. (II.60.7) What is the radius of the smallest circle that surrounds a 5-by-12 rectangle?

7. (II.42.11) In triangle *TOM*, let *P* be the midpoint of segment *TO* and let *Q* be the midpoint of segment *TM*. Draw the line through *P* parallel to segment *TM*, and the line through *Q* parallel to segment *TO*; these lines intersect at *J*. What can you say about the location of point *J*?

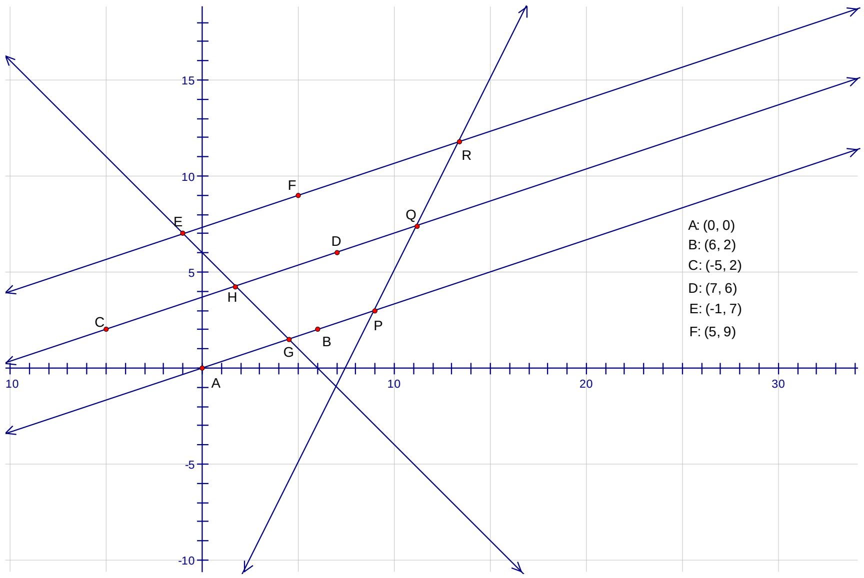
8. (II.38.9) The midpoints of the sides of a triangle are $(3, -1)$, $(4, 3)$, and $(0, 5)$. Find coordinates for the vertices of the triangle.

9. (II.43.8) A line drawn parallel to the side *BC* of triangle *ABC* intersects side *AB* at *P* and side *AC* at *Q*. The measurements $AP = 3.8$ in, $PB = 7.6$ in, and $AQ = 5.6$ in are made. If segment *QC* were now measured, how long would it be?

10. (II.79.6) One stick is three times as long as another, for example, one stick may measure 36 inches and the other 12 inches. You break the longer stick at a random point. Now you have three sticks. What is the probability that they form a triangle? Verify that the probability would be the same if the measures of the sticks were 51 inches and 17 inches.

11. (II.79.7) (Continuation) What is the probability that the three pieces form a triangle if the lengths of the original two sticks are *a* and *b*, and *a* is larger than *b*?

1. (II.42.9) Let $A = (0, 0)$, $B = (6, 2)$, $C = (-5, 2)$, $D = (7, 6)$, $E = (-1, 7)$, and $F = (5, 9)$. These points have been plotted below with lines *AB*, *CD*, and *EF*. Verify that they are parallel.
(a) The transversal of slope -1 that goes through *E* has also been drawn. It intersects line *AB* at *G* and line *CD* at *H*. Use your ruler to measure *EH* and *HG*.
(b) A transversal with slope 2 has been graphed, intersecting line *AB* at *P*, line *CD* at *Q*, and line *EF* at *R*. Use your ruler to measure *PQ* and *QR*.
(c) Investigate the ratios *EH:HG* and *RQ:QP*. On the basis of your findings, formulate the *Three Parallels Theorem*.



2. (II.43.4) Find coordinates for a point that is 5 units from the line $3x + 4y = 10$.

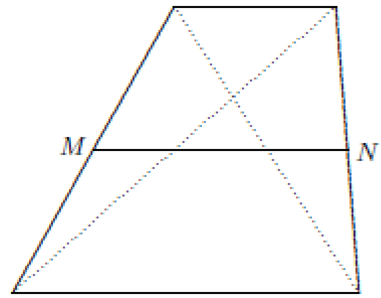
3. (II.44.11) Segments *AC* and *BD* intersect at *E*, so as to make *AE* twice *EC* and *BE* twice *ED*. Prove that segment *AB* is twice as long as segment *CD*, and parallel to it.

4. (II.50.1) In triangle *ABC*, points *M* and *N* are marked on sides *AB* and *AC*, respectively, so that $AM : AB = 17 : 100 = AN : AC$. Show that segments *MN* and *BC* are parallel.

5. (II.50.2) (Continuation) In triangle *ABC*, points *M* and *N* are marked on sides *AB* and *AC*, respectively, so that the ratios $AM : AB$ and $AN : AC$ are both equal to *r*, where *r* is some number between 0 and 1. Show that segments *MN* and *BC* are parallel.

6. (II.42.10) In triangle *ABC*, let *M* be the midpoint of *AB* and *N* be the midpoint of *AC*. Suppose that you measure *MN* and find it to be 7.3 cm long. How long would *BC* be, if you measured it? If you were to measure angles *AMN* and *ABC*, what would you find?

Problem-Based Mathematics II



1. (II.47.10) If M and N are the midpoints of the non-parallel sides of a trapezoid, it makes sense to call the segment MN the *midline* of the trapezoid. Why? (It actually should be called the *midsegment*, of course. Strange to say, some textbooks call it the *median*). Suppose that the parallel sides of a trapezoid have lengths 7 and 15. What is the length of the midline of the trapezoid? What are the lengths of the three pieces into which the midline is divided by the points where it intersects the diagonals of the trapezoid?

2. (III.1.2) In mathematical discussion, a *right prism* is defined to be a solid figure that has two parallel, congruent polygonal bases, and rectangular *lateral faces*. How would you find the volume of such a figure? Explain your method.

3. (II.44.8) Let $A = (0, 0)$, $B = (0, 21)$, and $C = (28, 0)$. Let F be the point where the bisector of angle BAC meets side BC. Find exact coordinates for F. Notice that F is not the midpoint of BC. Finally, calculate the distances BF and CF. Which of the two is larger and why do you think that is?

4. (II.44.9) (Continuation) Show that $AB{:}BF = AC{:}CF$. This is the *Angle Bisector Theorem.* Find another equivalent statement of the theorem using these side lengths.

5. (II.44.10) Choose four points for the vertices of a non-isosceles trapezoid $ABCD$, with AB longer than CD and parallel to CD. Extend AD and BC until they meet at E. According to the *Three Parallels Theorem*, what ratios will be equal? Verify that this is the case.

6. (III.1.7) In the middle of the nineteenth century, octagonal barns and sheds (and even some houses) became popular. How many cubic feet of grain would an octagonal barn hold if it were 12 feet tall and had a regular base with 10-foot edges?

7. (II.39.1) Suppose that square $PQRS$ has 15-cm sides, and that G and H are on QR and PQ, respectively, so that PH and QG are both 8 cm long. Let T be the point where PG meets SH. Find the size of angle STG, with justification.

8. (II.39.2) (Continuation) Find the lengths of PG and PT.

9. (II.46.11) Suppose that $ABCD$ is a trapezoid, with AB parallel to CD. Let M and N be the midpoints of DA and BC, respectively. What can be said about segment MN? Explain.

10. (II.48.14) The diagonals of a non-isosceles trapezoid divide the midline into three segments, whose lengths are 8 cm, 3 cm, and 8 cm. How long are the parallel sides? From this information, is it possible to infer anything about the distance that separates the parallel sides? Explain.

11. (II.66.9) Draw a circle with a 2-inch radius, mark four points randomly (not evenly spaced) on it, and label them consecutively G, E, O, and M. Measure angles GEO and GMO. Could you have predicted the result? Name another pair of angles that would have produced the same result.

1. (II.49.7) In trapezoid *ABCD*, *AB* is parallel to *CD*, and *AB* = 10, *BC* = 9, *CD* = 22, and *DA* = 15. Points *P* and *Q* are marked on *BC* so that *BP* = *PQ* = *QC* = 3, and points *R* and *S* are marked on *DA* so that *DR* = *RS* = *SA* = 5. Find the lengths *PS* and *QR*.

2. (II.46.7) Given triangle *ABC*, let *F* be the point where segment *BC* meets the bisector of angle *BAC*. Draw the line through *B* that is parallel to segment *AF*, and let *E* be the point where this parallel meets the extension of segment *CA*.
(a) Find the four congruent angles in your diagram.
(b) How are the lengths *EA*, *AC*, *BF*, and *FC* related?
(c) Use the above information to prove the *Angle Bisector Theorem*

3. (II.65.3) On a circle whose center is *O*, mark points *P* and *A* so that minor arc *PA* is a 46 degree arc. What does this tell you about angle *POA*?
Next, extend *PO* until it hits the circle. Call the point where this extension meets the circle *T*. Find angle *AOT*. Use this to find the size of angle *PTA*.

4. (II.65.4) (Continuation) If minor arc *PA* is a *k*-degree arc, what is the size of angle *PTA*? Justify your answer.

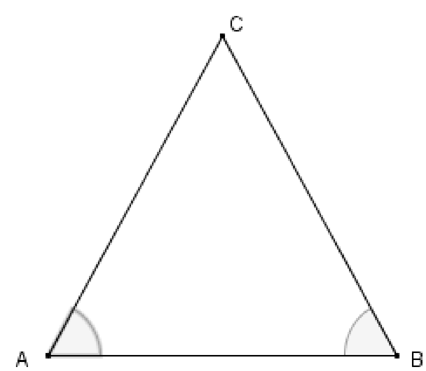
5. (SAT problem) A jar contains a red marble, a blue marble, and six green marbles. Alex draws one marble from the jar, and then Chris draws a marble from those remaining. What is the probability that Alex draws the red marble and Chris draws the blue marble?

6. (II.49.8) *The Varignon quadrilateral*. A quadrilateral has diagonals of lengths 8 and 10. The midpoints of the sides of this figure are joined to form a new quadrilateral. What is the perimeter of the new quadrilateral? What is special about it?

7. (II.49.10) The parallel bases of a trapezoid have lengths 12 and 18 cm. Find the lengths of the two segments into which the midline of the trapezoid is divided by a diagonal.

8. (II.47.1) In acute triangle *ABC*, the bisector of angle *ABC* meets side *AC* at *D*. Mark points *P* and *Q* on sides *BA* and *BC*, respectively, so that segment *DP* is perpendicular to *BA* and segment *DQ* is perpendicular to *BC*. Prove that triangles *BDP* and *BDQ* are congruent. What about triangles *PAD* and *QCD*?

9. (SAT Problem) If *ABC* (pictured at right) is an isosceles triangle with a height of 15 and base *AB* of length 16, find the perimeter of the triangle.



(II.45.1-45.17) In the following list of true statements, find **(a)** the statements whose converses are also in the list; **(b)** the statement that is a definition; **(c)** the statement(s) whose converse is false; **(d)** the Sentry Theorem; **(e)** the Midline Theorem; **(f)** the Three Parallels Theorem.

1. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is in fact a parallelogram.

2. If both pairs of opposite angles of a quadrilateral are congruent, then the quadrilateral must be a parallelogram.

3. If a quadrilateral is equilateral, then it is a rhombus.

4. If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

5. If a quadrilateral has two pairs of equal adjacent sides, then its diagonals are perpendicular.

6. If one of the medians of a triangle is half the length of the side to which it is drawn, then the triangle is a right triangle.

7. If a segment joins two of the midpoints of the sides of a triangle, then it is parallel to the third side, and is half the length of the third side.

8. Both pairs of opposite sides of a parallelogram are congruent.

9. The sum of the exterior angles of any polygon—one at each vertex—is 360 degrees.

10. The median drawn to the hypotenuse of a right triangle is half the length of the hypotenuse.

11. If two lines are intersected by a transversal so that alternate interior angles are equal, then the lines must be parallel.

12. The diagonals of a parallelogram bisect each other.

13. If two opposite sides of a quadrilateral are formed by equivalent vectors, then the quadrilateral is a parallelogram.

14. If three parallel lines intercept equal segments on one transversal, then they intercept equal segments on every transversal.

15. Both pairs of opposite angles of a parallelogram are congruent.

16. When a transversal intersects two parallel lines, the alternate interior angles are equal.

17. An exterior angle of a triangle is the sum of the two nonadjacent interior angles.

Problem-Based Mathematics II

1. (II.63.2) Draw a circle and label one of its diameters *AB*. Choose any other point on the circle and call it *C*. What can you say about the size of angle *ACB*? Does it depend on which *C* you chose? Justify your response.

2. (II.13.8) Find the number that is two thirds of the way (a) from −7 to 17; (b) from *m* to *n*.

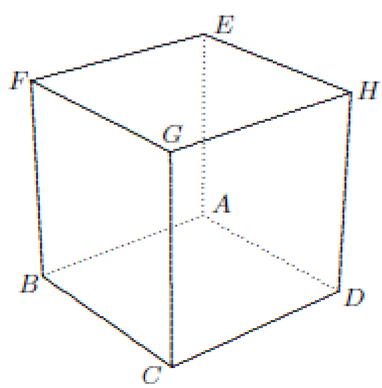
3. (II.48.5) A triangle, whose sides are 6, 8, and 10, and a circle, whose radius is *r*, are drawn so that no part of the triangle lies outside the circle. How small can *r* be?

4. (II.58.10) The vertices of triangle *ABC* are *A* = (−5, −12), *B* = (5, −12), and *C* = (5, 12). Confirm that the circumcenter of *ABC* lies at the origin. What is an equation for the circumcircle? Find the area of this circumcircle and also prove that angle ABC is right.

5. (I.48.8) Randy has 25% more money than Sandy, and 20% more money than Mandy, who has $1800. How much money does Sandy have?

6. (II.66.8) Given that triangle *ABC* is *similar* to triangle *PQR*, write the three term proportion that describes how the six sides of these figures are related.

7. (III.2.3) Given that *ABCDEFGH* is a cube (shown at right), what is significant about the square *pyramids ADHEG*, *ABCDG*, and *ABFEG*?



8. (III.81.4) Two distinct vertices of a cube are to be randomly chosen. Find the probability that the chosen vertices will be the endpoints of (a) an edge of the cube; (b) a *face diagonal* of the cube; (c) an *interior diagonal* of the cube.

9. (II.14.1) The components of vector [24, 7] are 24 and 7. Find the components of a vector that is three fifths as long as [24, 7].

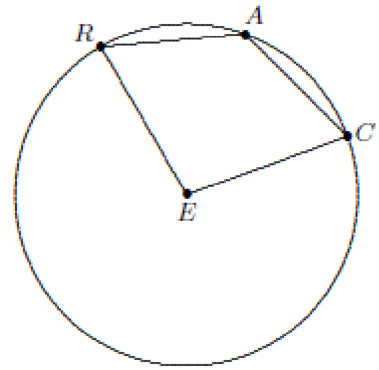
10. (II.69.11) If corresponding sides of two similar rectangles are in a 3:5 ratio, then what is the ratio of the:
(a) Perimeters of these rectangles?
(b) Areas of these rectangles?

11. (Continuation) If corresponding sides of two similar square pyramids are in a 3:5 ratio, then what is the ratio of their volumes?

Problem-Based Mathematics II

1. (II.14.2) Let $A = (-5, 2)$ and $B = (19, 9)$. Find coordinates for the point P between A and B that is three fifths of the way from A to B. Find coordinates for the point Q between A and B that is three fifths of the way from B to A.

2. (II.67.10) The figure at right shows points C, A, and R marked on a circle centered at E, so that *chords CA* and AR have the same length, and so that major arc CR is a 260-degree arc. Find the angles of quadrilateral $CARE$. What is special about the sizes of angles CAR and ACE?



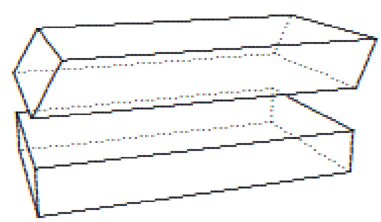
3. (III.2.8) The Great Pyramid at Gizeh was originally 483 feet tall, and it had a square base that was 756 feet on a side. It was built from rectangular stone blocks measuring 7 feet by 7 feet by 14 feet. Such a block weighs seventy tons. Approximately how many tons of stone were used to build the Great Pyramid? The volume of a pyramid is one third the base area times the height.

4. (II.67.1) A circular park 80 meters in diameter has a straight path cutting across it. It is 24 meters from the center of the park to the closest point on this path. How long is the path?

5. (II.16.1) The vector that is defined by a directed segment AB is often denoted $\overrightarrow{AB}$. Find components for the following vectors $\overrightarrow{AB}$:
(a) $A = (1, 2)$ and $B = (3, -7)$ (b) $A = (2, 3)$ and $B = (2 + 3t, 3 - 4t)$

6. (II.13.12) Let $K = (-2, 1)$ and $M = (3, 4)$. Find coordinates for the two points that divide segment KM into three congruent segments.

7. (III.1.8) Playing cards measure 2.25 inches by 3.5 inches. A full deck of fifty-two cards is 0.75 inches high. What is the volume of a deck of cards? If the cards were uniformly shifted (turning the bottom illustration into the top illustration), would this volume be affected?



8. (III.83.1) Three distinct vertices of a cube are to be randomly chosen. What is the probability that they will be the vertices of an equilateral triangle?

9. (II.20.7) Maintaining constant speed and direction for an hour, Whitney traveled from $(-2, 3)$ to $(10, 8)$. Where was Whitney after 35 minutes? What distance did Whitney cover in those 35 minutes?

10. (II.64.5) If two chords of a circle have the same length, then their minor arcs have the same length too. True or false? Explain. What about the converse statement? Is it true? Why?

1. (I.83.12) I have been observing the motion of a really tiny red bug on my graph paper. When I started watching, the bug was at the point (3, 4). Ten seconds later it was at (5, 5). Another ten seconds later it was at (7, 6). After another ten seconds it was at (9, 7).
(a) Draw a picture that illustrates what is happening.
(b) Write a description of any pattern that you notice. What assumptions are you making?
(c) Where was the bug 25 seconds after I started watching it?
(d) Where was the bug 26 seconds after I started watching it?

2. (I.84.11) From its initial position at (1, 6), an object moves linearly with constant speed. It reaches (7, 10) after two seconds and (13, 14) after four seconds.
(a) Predict the position of the object after six seconds; after nine seconds; after t seconds.
(b) Will there be a time when the object is the same distance from the x-axis as it is from the y-axis? If so, when, and where is the object?

3. (I.62.8) Give two examples of linear functions. Why are they called *linear*?

4. (I.86.6) The x- and y-coordinates of a point are given by the equations shown at right. The position of the point depends on the value assigned to t. Use your graph paper to plot points corresponding to the values $t = -4, -3, -2, -1, 0, 1, 2, 3$, and 4. Do you recognize any patterns? Describe what you see.

$$\begin{cases} x = 3 + t \\ y = 5 - 2t \end{cases}$$

5. (I.86.7) (Continuation) Plot the points (1, 2), (2, 5), and (3, 8) on the coordinate plane. Write equations, similar to those in the preceding exercise, that produce these points when t values are assigned. There is more than one correct answer.

6. (II.66.7) If P and Q are points on a circle, then the center of the circle must be on the perpendicular bisector of chord PQ. Explain. Which point on the chord is closest to the center? Why?

7. (II.8.4) The x- and y-coordinates of a point are given by the equations shown below. Use your graph paper to plot points corresponding to $t = -1$, 0, and 2. These points should appear to be collinear. Convince yourself that this is the case, and calculate the slope of this line. The displayed equations are called *parametric*, and t is called a *parameter*. How is the slope of a line determined from its parametric equations?

$$\begin{cases} x = -4 + 3t \\ y = 1 + 2t \end{cases}$$

8. (II.27.7) Find an equation for the line through point (7, 9) that is perpendicular to the vector [5, −2].

9. (II.8.2) Points (x, y) described by the equations $x = 1 + 2t$ and $y = 3 + t$ form a line. Is the point (7, 6) on this line? How about (−3, 1)? How about (6, 5.5)? How about (11, 7)?

1. (II.4.2) I have been observing the motion of a bug that is crawling on my graph paper. When I started watching, it was at the point (1, 2). Ten seconds later it was at (3, 5). Another ten seconds later it was at (5, 8). After another ten seconds it was at (7, 11).
(a) Draw a picture that illustrates what is happening.
(b)Write a description of any pattern that you notice. What assumptions are you making?
(c) Where was the bug 25 seconds after I started watching it?
(d)Where was the bug 26 seconds after I started watching it?

2. (II.4.10) A bug moves linearly with constant speed across my graph paper. I first notice the bug when it is at (3, 4). It reaches (9, 8) after two seconds and (15, 12) after four seconds.
(a) Predict the position of the bug after six seconds; after nine seconds; after t seconds.
(b) Is there a time when the bug is equidistant from the x- and y-axes? If so, where is it?

3. (I.49.4) Find the value of p that makes the linear graph $y = p - 3x$ pass through the point where the lines $4x - y = 6$ and $2x - 5y = 12$ intersect.

4. (II.5.6) The x- and y-coordinates of a point are given by the equations shown below. The position of the point depends on the value assigned to t. Use your graph paper to plot points corresponding to the values $t = -4, -3, -2, -1, 0, 1, 2, 3,$ and 4. Do you recognize any patterns? Describe them.

$$\begin{cases} x = 2 + 2t \\ y = 5 - t \end{cases}$$

5. (II.5.7) (Continuation) Plot the following points on the coordinate plane: (-1, 3), (0, 8), (2, 18). Write two different sets of parametric equations, similar to those in the preceding exercise, that produce these points when t values are assigned. There is more than one correct answer.

6. (II.68.3) Two circles of radius 10 cm are drawn so that their centers are 12 cm apart. The two points of intersection determine a *common chord*. Find the length of this chord.

7. (II.3.4) Una recently purchased two boxes of ten-inch candles—one box from a discount store, and the other from an expensive boutique. One evening, Una noticed that the inexpensive candles last only three hours each, while the expensive candles last five hours each. The next evening, Una hosted a dinner party and lit two candles—one from each box—at 7:30 pm. During dessert, a guest noticed that one candle was twice as long as the other. At what time was this observation made? Hint: This is similar to the tiny red bug problem.

8. (II.6.8) In a dream, Blair is confined to a coordinate plane, moving along a line with a constant speed. Blair's position at 4 am is (2, 5) and at 6 am it is (6, 3). What is Blair's position at 8:15 am when the alarm goes off?

9. (II.8.5) Find parametric equations to describe the line that goes through the points $A = (5,-3)$ and $B = (7, 1)$. There is more than one correct answer to this question.

10. (II.17.3) Is it possible for a line to go through (a) *no* lattice points? (b) exactly *one* lattice point? (c) exactly *two* lattice points? For each answer, either give an example or else explain the impossibility.

1. (II.9.1) Caught in another nightmare, Blair is moving along the line $y = 3x + 2$. At midnight, Blair's position is (1, 5), the *x*-coordinate increasing by 4 units every hour. Write parametric equations that describe Blair's position *t* hours after midnight. What was Blair's position at 10:15 pm when the nightmare started? Find Blair's speed, in units per hour.

2. (III.35.1) A cylinder of radius 4 and height *h* is inscribed in a sphere of radius 8. Find *h*.

3. (II.9.2) The parametric equations $x = -2 - 3t$ and $y = 6 + 4t$ describe the position of a particle, in meters and seconds. How does the particle's position change each second? each minute? What is the speed of the particle, in meters per second? Write parametric equations that describe this motion, using meters and *minutes* as units.

4. (II.10.2) A bug is moving along the line $3x + 4y = 12$ with constant speed 5 units per second. The bug crosses the *x*-axis when $t = 0$ seconds. It crosses the *y*-axis later. When? Where is the bug when $t = 2$? when $t = -1$? when $t = 1.5$? What does a negative *t*-value mean?

5. (II.10.4) Find parametric equations to describe the line $3x + 4y = 12$. Use your equations to find coordinates for the point that is three-fifths of the way from (4, 0) to (0, 3). By calculating some distances, verify that you have the correct point.

6. (II.27.6) Kim can run at 10 units per hour. The bank of a river is represented by the line $4x + 3y = 12$, and Kim is at (7, 5). How much time does Kim need to reach the river?

7. (I.87.3) The distance from here to downtown Pittsburgh is 10 miles. If you were to walk at a steady 4 mph, how long would it take you to complete the trip? If you were to ride your bike at 8 miles per hour, how long would it take you to complete the trip? Express the relationship between the speed and the time in an equation. How fast (miles per hour) must you travel if you want to complete this trip in 1 hour? in one minute? in one second?

8. (SAT problem) Let **$x** be defined by $\$x = \dfrac{x+3}{x-1}$ for any *x* not equal to 1. What is the value of \$9? How about \$7? Is \$(x-2) the same thing as (\$x) -2? Explain. Note that we usually use function notation to describe these types of definitions, as in $f(x) = \dfrac{x+3}{x-1}$.

9. (II.11.2) Kirby moves with constant speed 5 units per hour along the line $y = \frac{3}{4}x + 6$, crossing the *y*-axis at midnight and the *x*-axis later. When is the *x*-axis crossing made? What does it mean to say that *Kirby's position is a function of time*? What is Kirby's position 1.5 hours after midnight? What is Kirby's position *t* hours after midnight?

10. (II.12.1) A bug moves at 13 cm/sec, its position described by the parametric equations at right. Explain. Change the equations to obtain the description of a bug moving along the same line with speed 26 cm/second.

$$\begin{cases} x = 2 - 12t \\ y = 1 + 5t \end{cases}$$

1. (I.65.8) On a recent drive from Pittsburgh to Philadelphia, Taylor maintained an average speed of 50 mph for the first four hours, but could only average 30 mph for the final hour, because of road construction. What was Taylor's average speed for the whole trip? What would the average have been if Taylor had traveled x hours at 30 mph and $4x$ hours at 50 mph?

2. (II.9.5) Find parametric equations that describe the following lines:
(a) through (3, 1) and (7, 3)   (b) through (7,−1) and (7, 3)

3. (II.67.6) A chord 6 cm long is 2 cm from the center of a circle. How long is a chord that is 1 cm from the center of the same circle?

4. (II.24.1) An object moves with constant *velocity* (which means constant speed and direction) from (−3, 1) to (5, 7), taking five seconds for the trip.
(a) What is the speed of the object?
(b) Where does the object cross the y-axis?
(c) Where is the object three seconds after it leaves (−3, 1)?

5. (II.12.7) The initial position of an object is $P(0) = (7, -2)$. Its position after being displaced by the vector $t[-8, 7]$ is $P(t) = (7,-2) + t[-8, 7]$. Notice that the position is a function of t. Calculate $P(3)$, $P(2)$, and $P(-2)$. Describe the configuration of all possible positions $P(t)$.

6. (II.14.11) The lines defined by $P(t)=(4+5t,-1+2t)$ and $Q(u)=(4-2u,-1+5u)$ intersect perpendicularly. Justify this statement. What are the coordinates of the point of intersection?

7. (II.15.5) On the same coordinate-axis system, graph the line defined by $P(t)=(3t-4,2t-1)$ and the line defined by $4x+3y=18.$ The graphs should intersect in the first quadrant.
(a) Calculate $P(2)$, and show that it is not the point of intersection.
(b) Find the value of t for which $P(t)$ is on the line $4x+3y=18$.

8. (II.16.10) Motions of three particles are described by the following three pairs of equations

(a) $\begin{cases} x=2-2t \\ y=5+7t \end{cases}$ (b) $\begin{cases} x=4-2t \\ y=-2+7t \end{cases}$ (c) $\begin{cases} x=2-2(t+1) \\ y=5+7(t+1) \end{cases}$

How do the positions of these particles compare at any given moment?

9. (II.17.12) A particle moves according to $(x,y)=(6-t,-1+3t)$. For what value of t is the particle closest to the point (−2, 0)?

10. (II.17.14) What do the descriptions of motion defined by equations $P(t)=(-2+t,3+2t)$ and $Q(u)=(4-3u,-1-6u)$ have in common? How do they differ?

1. (SAT problem) For any positive integer n, let $n^{☺}$ be defined by $n^{☺} = 2n(n+1)$. What is the value of $8^{☺}$ divided by $2^{☺}$. Write the ☺ operation using traditional function notation.

2. (III.66.1) Morgan is going to roll a six-sided die five times. What is the probability that only the final roll will be a deuce?

3. (II.19.6) Graph the line that is described parametrically by $(x, y) = (2t, 4-t)$, then
(a) find the point on the line that minimizes the distance to (3, 8).
(b) confirm that the point corresponding to $t = 0$ is exactly 5 units from (3, 8);
(c) find the other point on the line that is 5 units from (3, 8);

4. (II.20.1) Let $A = (7, 7)$, $B = (5, 1)$, and $P(t) = (6+3t, 4-t)$. Plot A and B. Choose two values for t and plot the resulting points $P(t)$, which should look equidistant from A and B. Make calculations to confirm the equidistance.

5. (II.23.6) Brett and Jordan are out driving in the coordinate plane, each on a separate straight road. The equations $B(t) = (-3, 4) + t[1, 2]$ and $J(t) = (6, 1) + t[-1, 1]$ describe their respective travels, where t is the number of minutes after noon.
(a) Make a sketch of the two roads, with arrows to indicate direction of travel.
(b) Where do the two roads intersect?
(c) How fast is Brett going? How fast is Jordan going?
(d) Do they collide? If not, who gets to the intersection first?

6. (II.25.12) Find coordinates for the point where line $(x, y) = (3+2t, -1+3t)$ meets line $y = 2x-5$.

7. (II.26.6) What are the axis intercepts of the line described by $P(t) = (5+3t, -2 + 4t)$?

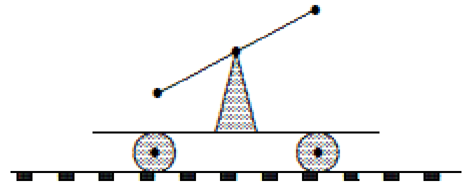
8. (II.28.2) Find the point of intersection of the lines given by $P(t) = (-1+3t, 3+2t)$ and $Q(r) = (4-r, 1+2r)$.

9. (II.24.5) Find the point on the y-axis that is equidistant from $A = (0, 0)$ and $B = (12, 5)$.

10. (I.50.5) A car went a distance of 90 km at a steady speed and returned along the same route at half that speed. Given that the time for the whole round trip was four and a half hours, find the two speeds.

11. (II.21.6) Let $A = (0, 0)$ and $B = (12, 5)$, and let C be the point on segment AB that is 8 units from A. Find coordinates for C.

1. (I.31.3) Pat and Kim are operating a handcar on the railroad tracks. It is hard work, and it takes an hour to cover each mile. Their big adventure starts at 8 am at Harmarville north of Aspinwall. They reach Main Street of Aspinwall at noon, and finish their ride in Etna at 3 pm. Let t be the number of hours since the trip began, and d be the corresponding distance (in miles) between the handcar and Main St. With t on the horizontal axis, draw a graph of d versus t, after first making a table of (t, d) pairs for $0 \le t \le 7$.



2. (I.31.4) (Continuation) Graph the equation $y = |x-4|$ for $0 \le x \le 7$. Interpret this graph in this context.

3. (I.31.5) (Continuation) Let y be the distance between the handcar and Fox Chapel Plaza, which Pat and Kim reach at 11 am. Draw a graph that plots y versus t, for the entire interval $0 \le t \le 7$. Write an equation that expresses y in terms of t. By the way, you have probably noticed that each of these absolute-value graphs has a corner point, which is called a *vertex*.

4. (I.31.6) (Continuation) Solve the equation $|x-3| = 1$ and interpret the answers.

5. (I.83.5) Evaluate $\sqrt{x^2+y^2}$ using x = 24 and y = 10. Is $\sqrt{x^2+y^2}$ equivalent to $x + y$? Does the square-root operation "distribute" over addition?

6. (I.83.6) Evaluate $\sqrt{(x+y)^2}$ using x = 24 and y = 10. Is $\sqrt{(x+y)^2}$ equivalent to $x + y$? Explain.

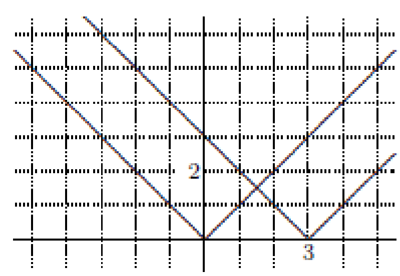
7. (I.83.7) Evaluate $\sqrt{(x+y)^2}$ using x = -24 and y = 10. Is $\sqrt{(x+y)^2}$ equivalent to $x + y$? Explain.

8. (I.88.3) Find the distance from $P = (3,\ 1)$ to $Q = (x,\ 1)$; from $P = (3,\ 1)$ to $Q = (x,\ y)$.

9. (I.33.10) Graph $y = |x| + 3$ and $y = |x| - 5$, then describe in general terms how the graph of $y = |x|$ is transformed to produce the graph of $y = |x| + k$. How can you tell from the graph whether k is positive or negative?

10. (I.32.12) Graph $y = |x-5|$ and $y = |x+3|$ then describe in general terms how the graph of $y = |x|$ is transformed to produce the graph of $y = |x-h|$.

11. (I.32.13) Write an equation for each of the graphs shown at right. Each graph goes through several *lattice points*.



1. (II.20.11) A particle moves according to the equation $(x, y) = (1, 2) + t[4, 3]$. Let P be the point where the path of this particle intersects the line $4x + 3y = 16$. Find coordinates for P, then explain why P is the point on $4x + 3y = 16$ that is closest to (1, 2).

2. (I.34.6) Find the x-and y-intercepts of $y = |x - 3| - 5$ and find the coordinates of its vertex.

3. (I.34.8) Sketch on the same axes the graphs of
(a) $y = |x|$ (b) $y = 2|x|$ (c) $y = 0.5|x|$ (d) $y = -3|x|$

4. (I.34.9) What effect does the coefficient a have on the graph of the equation $y = a|x|$? How can you tell whether a is positive or negative by looking at the graph?

5. (I.35.7) Compare the graphs of $y = x - 3$ and $y = |x - 3|$. How are they related?

6. (I.38.11) A hot-air balloon ride has been set up so that a paying customer is carried straight up at 50 feet per minute for ten minutes and then immediately brought back to the ground at the same rate. The whole ride lasts twenty minutes. Let h be the height of the balloon (in feet) and t be the number of minutes since the ride began. Draw a graph of h versus t. What are the coordinates of the vertex? Find an equation that expresses h in terms of t.

7. (I.30.3) The equation $|x - 7| = 2$ is a translation of "the distance from x to 7 is 2."
(a) Translate $|x - 7| \le 2$ into English, and graph its solutions on a number line.
(b) Convert "the distance from −5 to x is at most 3" into symbolic form, and solve it.

8. (I.45.4) What values of x satisfy the inequality $|x| > 12$? Graph this set on a number line, and describe it in words. Answer the same question for $|x - 2| > 12$.

9. (I.43.1) The fuel efficiency m (in miles per gallon) of a truck depends on the speed r (in miles per hour) at which it is driven. The relationship between m and r usually takes the form $m = a|r - h| + k$. For Sasha's truck, the optimal fuel efficiency is 24 miles per gallon attained when the truck is driven at 50 miles per hour. When Sasha drives at 60 miles per hour, however, the fuel efficiency drops to only 20 miles per gallon.

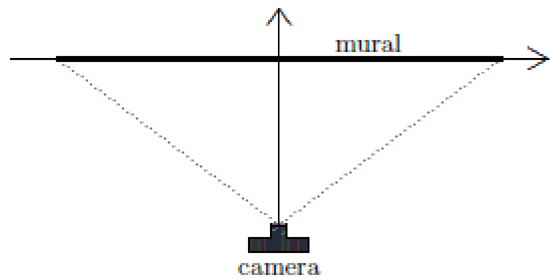
r	m
60	20
50	24
40	
30	
20	
10	

10. (I.39.3) My sleeping bag is advertised to be suitable for temperatures T between 20 degrees below zero and 20 degrees above zero (Celsius). Write an absolute-value inequality that describes these temperatures T.

11. (I.43.6) Graph $y = 2|x + 1| - 3$, then describe in general terms how the graph of $y = |x|$ is transformed to produce the graph of $y = a|x - h| + k$.

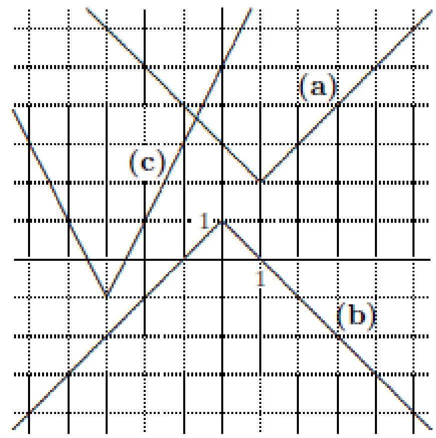
Problem-Based Mathematics II

1. (I.52.4) Using an absolute-value inequality, describe the set of numbers whose distance from 4 is greater than 5 units. Draw a graph of this set on a number line. Finally, describe this set of numbers using inequalities without absolute value signs.



2. (I.45.10) Using the coordinate-axis system shown in the top view at right, the viewing area of a camera aimed at a mural placed on the x-axis is bordered by $y = \frac{7}{8}|x| - 42$. The dimensions are in feet. How far is the camera from the x-axis, and how wide a mural can be photographed?

3. (I.46.6) Write a formula that expresses the distance between p and 17. Describe all the possible values for p if this distance is to be greater than 29.



4. (I.44.7) Write an equation for each of the graphs at right.

5. (I.40.5) Graph solutions on a number line:
(a) $|x+8| < 20$ (b) $|2x-5| \le 7$ (c) $3|4-x| \ge 12$

6. (I.47.5) Write an inequality that describes all the points that are more than 3 units from 5.

7. (I.48.1) My car averages 29 miles per gallon of gasoline, but I know—after many years of fueling it—that the actual miles per gallon can vary by as much as 3 either way. Write an absolute-value inequality that describes the range of possible mpg figures for my car.

8. (I.50.3) Impeded by the current, the Outing Club took 4 hours and 24 minutes to paddle 11 km up the Allegheny River to their campsite last weekend. The next day, the current was with them, and it took only 2 hours to make the return trip to campus. Everyone paddled with the same intensity on both days. At what rate would the paddlers have traveled if there had been no current? What was the speed of the current?

9. (I.53.6) On the same axes, sketch the graphs of $y = |x-3|$ and $y = 4 - |x-3|$. Label the points of intersection with coordinates. Find the area enclosed.

10. (I.60.8) Pat and Kim are having an algebra argument. Kim is sure that $-x^2$ is equivalent to $(-x)^2$, but Pat thinks otherwise. How would you resolve this disagreement? What evidence does your calculator offer?

1. (I.60.5) When an object falls, it gains speed. Thus the number of feet d the object has fallen is not linearly related to the number of seconds t spent falling. In fact, for objects falling near the surface of the Earth, with negligible resistance from the air, $d = 16t^2$. How many seconds would it take for a cannonball to reach the ground if it were dropped from the top of the Eiffel Tower, which is 984 feet tall? How many seconds would it take for the cannonball to reach the ground if it were dropped from a point that is halfway to the top?

2. (I.62.2) Complete the table at right. Then graph by hand on separate axes $y = |x|$ and $y = x^2$. Check your graphs with your calculator. In what respects are the two graphs similar? In what respects do the two graphs differ?

x	$\lvert x\rvert$	x^2
-2		
-1		
$-1/2$		
0		
$1/2$		
1		
2		

3. (I.41.8) Graph $y = 3|x-2| - 6$, and find coordinates for the vertex and the x-and y-intercepts.

4. (I.77.5) The three functions $y = 2(x-4) - 1$, $y = 2|x-4| - 1$, and $y = 2(x-4)^2 - 1$ look somewhat similar. Predict what the graph of each will look like, and then sketch them in your notebook (without using a calculator) by just plotting a few key points. In each case think about how the form of the equation can help provide information.

5. (IV.7.1) When two six-sided dice both land showing ones, this is called *snake-eyes*. What is the probability of this happening? What is the most likely sum of two dice?

6. (I.61.3) An avid gardener, Gerry Anium just bought 80 feet of decorative fencing to create a border around a new rectangular garden that is still being designed.

width	*length*	*area*
5		
9		
16		
22		
24		
35		
x		

(a) If one of the sides of the rectangle were 5 feet long, what would the lengths of the other three sides and the enclosed area be? Write this data in the first row of the table.
(b) Record data for the next five examples in the table.
(c) Let x be the width of the garden. In terms of x, fill in the last row of the table.
(d) Use your calculator to graph the rectangle's area versus x, for $0 \le x \le 40$. As a check, you can make a scatter plot using the table data. What is special about the values $x = 0$ and $x = 40$?
(e) Comment on the symmetric appearance of the graph. Why was it predictable?
(f) Find the point on the graph that corresponds to the largest rectangular area that Gerry can enclose using the 80 feet of available fencing. This point is called the *vertex*.

7. (I.69.5) Quinn is four miles from the train station, from which a train is due to leave in 56 minutes. Quinn is walking along at 3 mph, and could run at 12 mph if it were necessary. If Quinn wants to be on that train, it will be necessary to do some running! How many miles of running?

8. (I.65.2) Sketch the graphs of $y = x^2 + 5$, $y = x^2 - 4$, and $y = x^2 + 1$ on the same axes. What is the effect of the value of c in equations of the form $y = x^2 + c$?

1. (SAT problem) A Monopoly player rolls two standard 6-sided dice and is astonished to discover that both individual dice show prime numbers, and their sum is also a prime number. What is the probability of this outcome?

2. (I.62.4) A workman accidentally drops a hammer from the scaffolding of a tall building. The workman is 300 feet above the ground. As you answer the following, recall that an object falls $16t^2$ feet in t seconds (assuming negligible air resistance).
(a) How far above the ground is the hammer after falling for one second? for two seconds? Write a formula that expresses the height h of the hammer after it has fallen for t seconds.
(b) How many seconds does it take the hammer to reach the ground? How many seconds does it take for the hammer to fall until it is 100 feet above the ground?
(c) By plotting some data points and connecting the dots, sketch a graph of h versus t. Notice that your graph is not a picture of the path followed by the falling hammer.

3. (I.62.6) Equations such as $A = 40x - x^2$ and $h = 300 - 16t^2$ define *quadratic functions*. The word *function* means that assigning a value to one of the variables (x or t) determines a *unique* value for the other (A or h). It is customary to say that "A is a function of x." In this example, however, it would be incorrect to say that "x is a function of A." Explain.

4. (I.62.7) The graph of a quadratic function is called a *parabola*. This shape is common to all graphs of equations of the form $y = ax^2 + bx + c$, where a is nonzero. Confirm this by comparing the graph of $y = x^2$, the graph of $y = 40x - x^2$ and the graph of $y = 300 - 16x^2$. How are the three graphs alike, and how are they different? What graphing window do you need to make sure all vertices and intercepts are displayed?

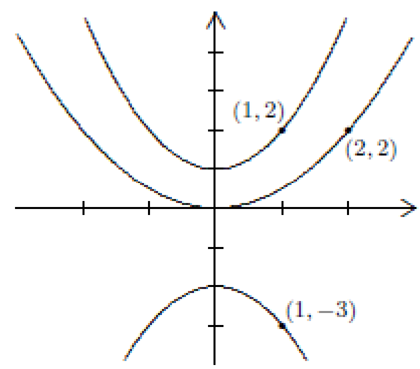
5. (I.65.5) Graph the equations on the same system of axes: $y = x^2$, $y = 0.5x^2$, $y = 2x^2$, and $y = -x^2$. What is the effect of a in equations of the form $y = ax^2$?

6. (I.66.1) Near the surface of the earth, assuming negligible resistance from the air, the height in feet of a falling object is modeled well by the equation $y = h - 16t^2$, where y is the height of the object, t is the number of seconds the object has been falling, and h is the height from which the object is dropped.
(a) If an iron ball were dropped from the Washington Monument, which is 555 feet high, how far above the ground would the ball be after 2 seconds of falling? How long would it take for the ball to hit the ground?
(b) Due to air resistance, a falling bag of corn chips will not gain speed as rapidly as a falling iron ball. Cal Elayo, a student of science, found that the descent of a falling bag of chips is modeled well by the equation $y = h - 2.5t^2$. In an historic experiment, Cal dropped a bag of chips from a point halfway up the Monument, while a friend simultaneously dropped the iron ball from the top. After how many seconds did the ball overtake the bag of chips?
(c) Graph the equations $y = 277.5 - 2.5t^2$ and $y = 555 - 16t^2$ on the same system of axes. Calculate the y-and t-intercepts of both curves. What is the meaning of these numbers? Notice that the curves intersect. What is the meaning of the intersection point?

1. (I.59.4) It would take Tom 8 hours to whitewash the fence in the backyard. His friend Huck would need 12 hours to do the same job by himself. They both start work at 9 in the morning, each at opposite ends of the fence, under the watchful eye of Tom's Aunt Polly. At what time in the afternoon is the task complete?

2. (I.66.3) For the point (4, 24) to be on the graph of $f(x) = ax^2$, what should the value of a be?

3. (I.68.2) Find a *quadratic equation* for each of the graphs pictured at the right. Each curve has a designated point on it, and the y-intercepts are all at integer values. Also notice that for each parabola, the y-axis is the *axis of symmetry* – the line about which the parabola is symmetric.



4. (I.68.4) The point (4, 7) is on the graph of $y = x^2 + c$. What is the value of c?

5. (SAT problem) If $x \uparrow y$ is defined as $x + y^2$ then find $3\uparrow 5$. Is this equal to $5\uparrow 3$? When does $x\uparrow y = y\uparrow x$?

6. (I.71.3) Sketch the graphs of $y = x^2$, $y = (x-2)^2$, $y = (x+3)^2$, and $y = (x-5)^2$ on the same set of coordinate axes. Make a general statement as to how the graph of $y = (x-h)^2$ is related to the graph of $y = x^2$.

7. (I.71.4) (Continuation) Sketch the graphs $y = 2(x-3)^2$, $y = -3(x-3)^2$, and $y = 0.5(x-3)^2$. What do these graphs all have in common? How do they differ? What is the equation of a parabola whose vertex is at the point (−2, 0), is the same size as the graph $y = 2(x-3)^2$, and opens up?

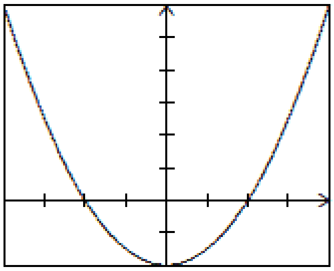
8. (II.71.12) Find an equation for the circle of radius 5 whose center is at (3,−1).

9. (I.71.6) The axis of symmetry of a parabola is the line $x = 4$.
(a) Suppose that one x-intercept is 10; what is the other one?
(b) Suppose the point (12, 4) is on the graph; what other point also must be on the graph?

10. (I.72.4) Sketch the graphs of $y = (x-4)^2$ and $y = 9$ on your calculator screen. What are the coordinates of the point(s) of intersection? Now solve the equation $(x-4)^2 = 9$. Describe the connection between the points of intersection on the graph and the solution(s) to the equation.

Problem-Based Mathematics II

1. (I.73.1) The graph of $f(x) = x^2 - 400$ is shown at right. Notice that no coordinates appear in the diagram. There are tick marks on the axes, however, which enable you, without using your graphing calculator, to figure out the actual window that was used for this graph. Find the high and low values for both the *x*-axis and the *y* axis. After you get your answer, check it on your calculator. To arrive at your answer, did you actually need to have tick marks on *both* axes?

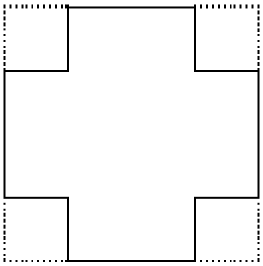


2. (I.73.2) Sketch the graph of $y = x^2 + 3$ and $y = |x| + 3$ on the same axis in your notebook. List three ways that the two graphs are alike and three ways in which they differ. Be sure your graph is large enough to clearly show these differences. On another axis, sketch the graph of $y = 2(x-3)^2$ and $y = 2|x-3|$. Also be prepared to explain how these two graphs compare.

3. (I.74.6) Graph the equations $y = (x-5)^2$, $y = (x-5)^2 - 4$, and $y = (x-5)^2 + 2$. Write the coordinates of the vertex for each curve. Describe how to transform the first parabola to obtain the other two. Next, a fourth parabola is created by shifting the first parabola so that its vertex is (5, −7). Write an equation for the fourth parabola.

4. (I.76.2) Sketch the graphs of $y = (x-4)^2$ and $y = (4-x)^2$. What do you notice about the graphs? Explain why this is true.

5. (I.76.5) The diagram at the right suggests an easy way of making a box with no top. Start with a square piece of cardboard, cut squares of equal sides from the four corners, and then fold up the sides. Here is the problem: To produce a box that is 8 cm deep and whose capacity is exactly one liter (1000 cc). How large a square must you start with (to the nearest mm)?



6. (I.47.2) Working alone, Jess can rake the leaves off a lawn in 50 minutes. Working alone, cousin Tate can do the same job in 30 minutes. Today they are going to work together, Jess starting at one end of the lawn and Tate starting simultaneously at the other end. In how many minutes will they meet and thus have the lawn completely raked?

7. (I.47.3) (Continuation) Suppose that Tate takes a ten-minute break after just five minutes of raking. Revise your prediction of how many minutes it will take to complete the job.

8. (I.77.6) Without using a calculator, make a sketch of the parabola $y = (x-50)^2 - 100$, by finding the *x*-intercepts, the *y*-intercept, and the coordinates of the vertex. Label all four points with their coordinates on your graph.

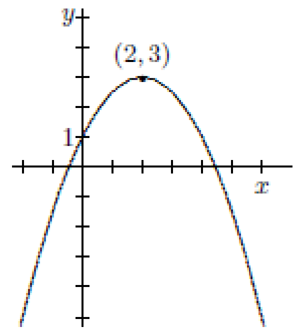
Problem-Based Mathematics II

1. (I.78.8) Find the x-intercepts of the following graphs, without expanding each squared term:

(a) $y=(x-4)^2-9$ (b) $y=-2(x+3)^2+8$

Check your work by sketching each parabola, incorporating the vertex and x-intercepts.

2. (I.80.1) When asked to find the equation of the parabola pictured at right, Ryan reasoned that the correct answer had to look like $y=a(x-2)^2+3$, for some value of a. Justify Ryan's reasoning, then finish the problem by finding the correct value of a.



3. (I.80.2) (Continuation) Find an equation for the parabola whose symmetry axis is parallel to the y-axis, whose vertex is $(-1, 4)$, and whose graph contains the point $(1, 3)$.

4. (II.19.5) Find a point on the line $2x+y=8$ that is equidistant from the points $(3, 8)$ and $(9, 6)$.

5. (II.73.10) Write an equation for the circle that is centered at $(-4, 5)$ and tangent to the x-axis. Find the area and circumference of this circle.

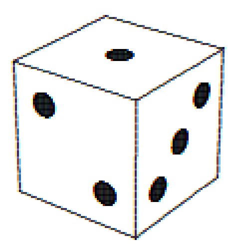
6. (I.81.2) Graph the equation $y=(x-5)^2-7$ without a calculator by plotting its vertex and its x-intercepts (just estimate their positions between two consecutive integers). Then use your calculator to draw the parabola. Repeat the process on $y=-2(x+6)^2+10$

7. (I.85.4) Find the x-intercepts in exact form of each of the following graphs:
(a) $y=(x-6)^2-10$ (b) $y=3(x-7)^2-9$ (c) $y=120-3x^2$ (d) $y=4.2-0.7x^2$

8. (II.23.11) Find coordinates for a point on the line $4y = 3x$ that is 8 units from $(0, 0)$.

9. (I.81.3) At most how many solutions can a quadratic equation have? Give an example of a quadratic equation that has two solutions. Give an example of a quadratic equation that has only one solution. Give an example of a quadratic equation that has no **real** solutions.

10. (III.85.1) Una is going to roll ten standard (six-sided) dice, one after another. What is the probability that
(a) *none* of the dice land showing an ace (a single spot) on top?
(b) *some* (at least one) of the dice land showing an ace on top?
(c) the first die shows an ace, but none of the others do?
(d) the last die shows an ace, but none of the others do?
(e) *exactly one* of the ten dice shows an ace on top?



1. (I.64.9) Graph the following parabolas with the given windows:
(a) $y = 0.001x^2$ in the window $-1000 \le x \le 1000$ and $0 \le y \le 1000$
(b) $y = 0.01x^2$ in the window $-100 \le x \le 100$ and $0 \le y \le 100$.
What do you notice about these two graphs?

2. (I.64.9) (Continuation) Avery and Alden were comparing parabola graphs on their calculators. Avery had drawn $y = 0.001x^2$ in the window $-1000 \le x \le 1000$ and $0 \le y \le 1000$, and Alden had drawn $y = x^2$ in the window $-k \le x \le k$ and $0 \le y \le k$. Except for scale markings on the axes, the graphs looked exactly the same! What was the value of k?

3. (I.88.12) A rectangle has an area of 36 square meters. Its length is $2\sqrt{3}$ meters. In exact form, what is the perimeter of the rectangle?

4. (I.68.5) In your notebook, use one set of coordinate axes to graph the three curves $y = x^2 - x,$ $y = x^2 + 2x,$ and $y = x^2 - 4x.$ Make three observations about graphs of the form $y = x^2 + bx,$ where b is a nonzero number.

5. (I.63.2) Factor each of the following quadratic expressions:
(a) $x^2 + 4x$ (b) $2x^2 - 6x$ (c) $3x^2 - 15x$ (d) $-2x^2 - 7x$

6. (I.63.3) (Continuation) The *zero-product property* says that $a \cdot b = 0$ is true if $a = 0$ or $b = 0$ is true, and only if $a = 0$ or $b = 0$ is true. Explain this property in your own words (looking up the word or in the Reference section if necessary). Apply it to solve these equations:
(a) $x^2 + 4x = 0$ (b) $2x^2 - 6x = 0$ (c) $3x^2 - 15x = 0$ (d) $-2x^2 - 7x = 0$

7. (I.63.4) (Continuation) Find the x-intercepts of each of the following quadratic graphs:
(a) $y = x^2 + 4x$ (b) $y = 2x^2 - 6x$ (c) $y = 3x^2 - 15x$ (d) $y = -2x^2 - 7x$
Summarize by describing how to find the x-intercepts of any quadratic graph $y = ax^2 + bx$.

8. (I.70.1) The height h (in feet) above the ground of a baseball depends upon the time t (in seconds) it has been in flight. Cameron takes a mighty swing and hits a bloop single whose height is described approximately by the equation $h = 80t - 16t^2$. Without resorting to graphing on your calculator, answer the following questions:
(a) How long is the ball in the air?
(b) The ball reaches its maximum height after how many seconds of flight?
(c) What is the maximum height?
(d) It takes 0.92 seconds for the ball to reach a height of 60 feet. On its way back down, the ball is again 60 feet above the ground; what is the value of t when this happens?

9. (II.21.5) A rhombus has 25-cm sides, and one diagonal is 14 cm long. How long is the other diagonal?

Problem-Based Mathematics II

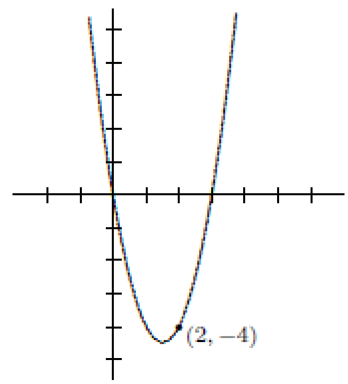
1. (I.70.3) Solve the following equations for x without using a calculator:
(a) $x^2 - 5x = 0$ (b) $3x^2 + 6x = 0$ (c) $ax^2 + bx = 0$

2. (I.70.6) Sketch the graphs of $y = x^2 - 12x$, $y = -2x^2 - 14x$, and $y = 3x^2 + 18x$. Write an equation for the symmetry axis of each parabola. Devise a quick way to write an equation for the symmetry axis of any parabola $y = ax^2 + bx$. Test your method on the three given examples

3. (II.62.10) The vertices of a square with sides parallel to the coordinate axes lie on the circle of radius 5 whose center is at the origin. Find coordinates for the four vertices of this square.

4. (I.73.6) There are several quadratic functions whose graphs intersect the x-axis at (0, 0) and (6, 0). Sketch graphs for a few of them, including the one that goes through (3, 9). Other than their axis of symmetry, what do all these graphs have in common? How do the graphs differ?

5. (I.73.4) When asked to find the equation of the parabola pictured at right, Ryan took one look at the x-intercepts and knew that the answer had to look like $y = ax(x-3)$, for some value of a. Justify Ryan's reasoning, then finish the problem by finding the correct value of a.

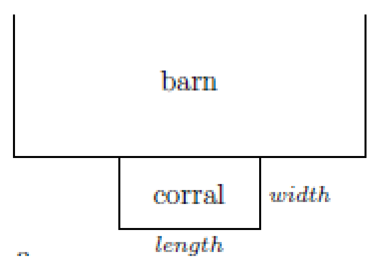


6. (I.73.5) (Continuation) Find an equation for the parabola whose symmetry axis is parallel to the y-axis, whose x-intercepts are −2 and 3, and whose y-intercept is 4.

7. (I.77.8) The graph of a quadratic function intersects the x-axis at 0 and at 8. Draw two parabolas that fit this description and find equations for them. How many examples are possible?

8. (I.77.9) Find an equation for the parabola whose x-intercepts are 0 and 8, whose axis of symmetry is parallel to the y-axis, and whose vertex is at
(a) (4, −16) (b) (4, −8) (c) (4, −4) (d) (4, 16)

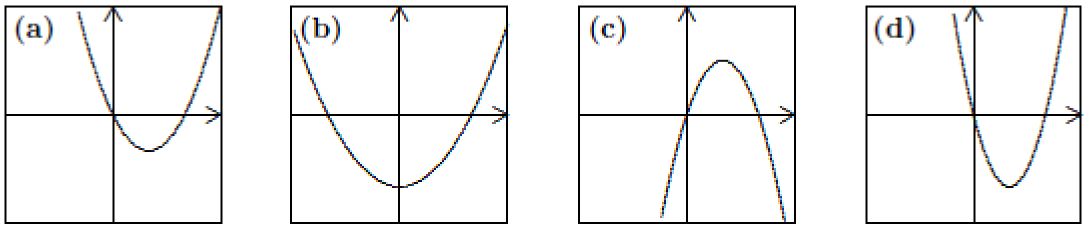
9. (I.82.5) Sam breeds horses, and is planning to construct a rectangular corral next to the barn, using a side of the barn as one side of the corral. Sam has 240 feet of fencing available, and has to decide how much of it to use for the width of the corral.
(a) Suppose the width is 50 feet. What is the length? How much area would this corral enclose?
(b) Suppose the width is 80 feet. What is the enclosed area?
(c) Suppose the width is x feet. Express the length and the enclosed area in terms of x.



10. (I.82.6) (Continuation) Let y stand for the area of the corral that corresponds to width x. Notice that y is a quadratic function of x. Sketch a graph of y versus x. For what values of x does this graph make sense? For what value of x does y attain its largest value? What are the dimensions of the corresponding corral?

Problem-Based Mathematics II

1. (I.82.8) Which of the following calculator screens could be displaying the graph of $y = x^2 - 2x$?

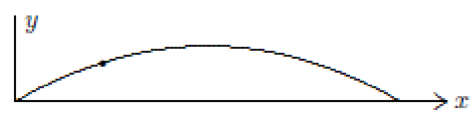


2. (II.14.12) What number is exactly midway between $23 - \sqrt{17}$ and $23 + \sqrt{17}$? What number is exactly midway between $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$?

3. (I.78.3) For each of the following, first write down the factors of the given expression. Then graph the function and record the *x*-intercepts.
(a) Factor $x^2 - 4x - 5 = 0$ and then graph $y = x^2 - 4x - 5$.
(b) Factor $x^2 + 12x + 35 = 0$ and then graph $y = x^2 + 12x + 35$.
(c) Factor $x^2 - 3x + 2 = 0$ and then graph $y = x^2 - 3x + 2$.
What is the connection between factoring a polynomial and finding the *x*-intercepts of its graph? Explain how the graph of a polynomial can help in the search for factors.

4. (I.78.4) (Continuation) Find the solutions to the equations
(a) $x^2 - 4x - 5 = 0$ (b) $x^2 + 12x + 35 = 0$ (c) $x^2 - 3x + 2 = 0$
Explain the reasoning used to solve a polynomial equation that is in factored form.

5. (I.63.7) *Golf math I*. Using a driver on the 7th tee, Dale hits an excellent shot, right down the middle of the level fairway. The ball follows the parabolic path shown in the figure, described by the quadratic function $y = 0.5x - 0.002x^2$, which relates the height *y* of the ball above the ground to the ball's progress *x* down the fairway. Distances are measured in yards.

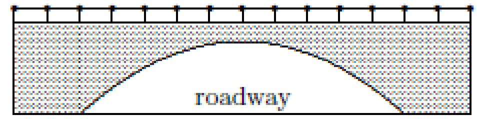


(a) Use the distributive property to write this equation in factored form. Notice that *y* = 0 when *x* = 0. What is the significance of this data?
(b) How far from the tee does the ball hit the ground?
(c) At what distance *x* does the ball reach the highest point of its arc? What is the maximal height attained by the ball?

6. (SAT problem) What is the probability of flipping three coins and having two of them come up heads and one come up tails?

7. (I.91.3) On a single set of coordinate axes, graph several parabolas of the form $y = bx - x^2$. Mark the vertex on each curve. What do you notice about the configuration of all such vertices?

1. (I.86.4) The figure shows a bridge arching over the Parkway East. To accommodate the road beneath, the arch is 100 feet wide at its base, 20 feet high in the center, and parabolic in shape. (a) The arch can be described by $y = kx(x - 100)$, if the origin is placed at the left end of the arch. Find the value of the coefficient k that makes the equation fit the arch.



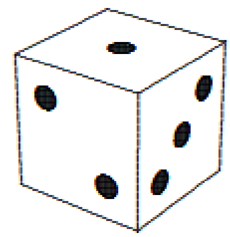
(b) Is it possible to move a rectangular object that is 40 feet wide and 16.5 feet high (a wide trailer, for example) through the opening? Explain.

2. (I.82.7) In each of the following, supply the missing factor:

(a) $2x^2 - 5x - 12 = (2x+3)(\qquad)$ (b) $3x^2 - 2x - 1 = (3x+1)(\qquad)$

(c) $4y^2 - 8y + 3 = (2y-1)(\qquad)$ (d) $6t^2 - 7t - 3 = (3t+1)(\qquad)$

3. (III.64.11) A six-sided die is to be rolled once.
(a) What is the probability of obtaining a "2" (called a *deuce*)?
(b) What is the probability of obtaining a non-deuce?

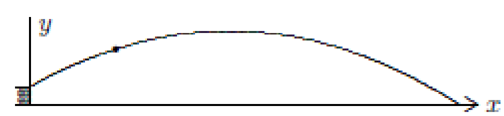


Next, the same die is to be rolled three times.
(c) What is the probability of obtaining three deuces?
(d) three non-deuces?
(e) exactly one deuce?
(f) at least one deuce?

4. (I.71.10) Find the equation of the axis of symmetry for the graph of $y = 2x^2 - 6x$. Sketch the graph, including the axis of symmetry. What are the coordinates of the vertex of the graph?

5. (I.71.11) (Continuation) Sketch the graph of $y = 2x^2 - 6x - 3$ along with its axis of symmetry. Find the coordinates of the vertex of this parabola. How do these coordinates compare with the vertex of $y = 2x^2 - 6x$? Find an equation for the graph of a quadratic curve that has the same axis of symmetry as $y = 2x^2 - 6x$, but whose vertex is at (1.5, −2.5).

6. (I.65.3) *Golf math II.* Again using a driver on the 8th tee, which is on a plateau 10 yards above the level fairway, Dale hits another fine shot. Explain why the quadratic function $y = 10.0 + 0.5x - 0.002x^2$ describes this parabolic trajectory, shown in the figure above. Why should you expect this tee shot to go more than 250 yards? Estimate the length of this shot, then use your calculator to find a more accurate value. How does this trajectory relate to the trajectory for the drive on the previous hole?



7. (I.65.4) (Continuation) To find the length of the shot without a calculator, you must set y equal to 0 and solve for x. Explain why, and show how to arrive at $x^2 - 250x = 5000$.
(a) The next step is to add 125^2 to both sides of this equation. Why was this number chosen?
(b) Complete the solution by showing that the length of the shot is $125 + \sqrt{20625}$. How does this number, which is about 268.6, compare with your previous calculation?
(c) Comment on the presence of the number 125 in the answer. What is its significance?

Problem-Based Mathematics II

1. (II.60.4) Sketch the circle whose equation is $x^2 + y^2 = 100$. Using the same system of coordinate axes, graph the line $x + 3y = 10$, which should intersect the circle twice: at $A = (10, 0)$ and at another point B in the second quadrant. Estimate the coordinates of B. Now use algebra to find them exactly.

2. (I.66.2) Sketch a graph for each of the following quadratic functions. Identify the coordinates of each vertex and write an equation for each axis of symmetry.
(a) $y = 3x^2 + 6$ (b) $y = x^2 + 6x$ (c) $y = 64 - 4x^2$ (d) $y = x^2 - 2x - 8$

3. (I.72.1) The table at right displays some values for a quadratic function $f(x) = ax^2 + bx + c$.

x	0	1	2	3	4
$f(x)$	0	2	6	12	20

(a) Explain how to use the table to show that $c = 0$.
(b) A point is on a curve only if the coordinates of the point satisfy the equation of the curve. Substitute the tabled coordinates (1, 2) into the given equation to obtain a linear equation in which a and b are the unknowns. Apply the same reasoning to the point (2, 6).
(c) Find values for a and b by solving these two linear equations.
(d) Use your values for a and b to identify the original quadratic equation. Check your result by substituting the other two tabled points (3, 12) and (4, 20) into the equation.

4. (I.74.1) In solving an equation such as $2x^2 - 5x = 3$ by completing the square, it is customary to first divide each term by 2 so that the coefficient of is x^2 is 1. This transforms the equation into $x^2 - \frac{5}{2}x = \frac{3}{2}$. Now continue to solve by the completing the square method, remembering to take half of $\frac{5}{2}$, square it and add it to *both* sides of the equation. Complete the solution.

5. (I.74.2) *Completing the square*. Confirm that the equation $ax^2 + bx + c = 0$ can be converted into the form $x^2 + \frac{b}{a}x = -\frac{c}{a}$. Describe the steps. To achieve the goal suggested by the title, what should now be added to both sides of this equation?

6. (I.74.3) (Continuation) The left side of the equation $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$ can be factored as a perfect square trinomial. Show how. The right side of the equation can be combined over a common denominator. Show how. Finish the solution of the general quadratic equation by taking the square root of both sides of your most recent equation. The answer is called *the quadratic formula*. You may be able to check your formula using your calculator. Apply your formula: Solve $x^2 + 2x - 3 = 0$ by letting $a = 1$, $b = 2$, and $c = -3$.

7. (I.84.1) I am thinking of a right triangle, whose sides can be represented by $x - 5$, $2x$, and $2x + 1$. Find the lengths of the three sides. Note: Solving your equation leads to answers that do not fit the context of the problem and therefore should be discarded.

1. (I.74.4) As long as the coefficients *a* and *b* are nonzero, the parabolic graph $y = ax^2 + bx$ has two *x*-intercepts. What are they? Use them to find the axis of symmetry for this parabola. Explain why the axis of symmetry for $y = 2x^2 - 5x - 12$ is the same as the axis of symmetry for $y = 2x^2 - 5x.$ In general, what is the symmetry axis for $y = ax^2 + bx + c$? Does your description make sense for $y = 2x^2 - 5x + 7,$ even though the curve has no *x*-intercepts?

2. (I.74.5) (Continuation) If you know the axis of symmetry for a quadratic function, how do you find the coordinates of the vertex? Try your method on each of the following, by first finding the symmetry axis, then the coordinates of the vertex.

(a) $y = x^2 + 2x - 3$ (b) $y = 3x^2 + 4x + 1$

3. (I.69.6) The work at right shows the step-by-step process used by a student to solve $x^2 + 6x - 5 = 0$ by the method of completing the square. Explain why the steps in this process are reversible. Apply this understanding to find a quadratic equation $ax^2 + bx + c = 0$ whose solutions are $x = 7 + \sqrt{6}$ and $x = 7 - \sqrt{6}.$

$$x^2 + 6x - 5 = 0$$
$$x^2 + 6x + 9 = 5 + 9$$
$$(x+3)^2 = 14$$
$$x + 3 = \pm\sqrt{14}$$
$$x = -3 \pm \sqrt{14}$$

4. (I.87.6) A mathematics teacher wants to make up a quadratic equation $ax^2 + bx + c = 0$, so that *a*, *b*, and *c* are integers, and the correct solutions are $x = \frac{1}{2}$ and $x = -3$. Find values for *a*, *b*, and *c* that will do the job. Is there more than one equation that will work?

5. (I.85.9) Expand and simplify:

(a) $(x-1)(x+1)$ (b) $(x-1)(x^2+x+1)$ (c) $(x-1)(x^3+x^2+x+1)$

6. (I.86.1) (Continuation) Write $x^5 - 1$ as the product of two factors.

7. (I.86.3) Factor each of the following as completely as you can:

(a) $p^4 - 4p^2$ (b) $w^3 - 2w^2 - 15w$ (c) $16y - 9yz^2$ (d) $2x^2 + 20x + 50$

8. (I.77.11) Solve $x^2 + bx + c = 0$ by the method of completing the square. Apply your answer to the example $x^2 + 5x + 6 = 0$ by setting $b = 5$ and $c = 6$.

9. (I.70.2) Apply the zero-product property to solve the following equations:

(a) $(x-2)(x+3) = 0$ (b) $x(2x+5) = 0$ (c) $5(x-1)(x+4)(2x-3) = 0$

10. (II.49.4) A rectangle is 2 inches wide, and more than 2 inches long. It so happens that this rectangle can be divided, by a single cut, into a 2-inch square and a small rectangle that has exactly the same shape as the large rectangle. What is the length of the large rectangle?

1. (I.69.2) A hose used by the fire department shoots water out in a parabolic arc. Let x be the horizontal distance from the hose's nozzle, and y be the corresponding height of the stream of water, both in feet. The quadratic function is $y = -0.016x^2 + 0.5x + 4.5$.
(a) What is the significance of the 4.5 that appears in the equation?
(b) Use your calculator to graph this function. Find the stream's greatest height.
(c) What is the horizontal distance from the nozzle to where the stream hits the ground?
(d) Will the stream go over a 6-foot high fence that is located 28 feet from the nozzle? Explain your reasoning.

2. (I.79.2) Find at least three integers (positive or negative) that, when put in the blank space, make the expression $x^2 + \_\_\_\ x - 36$ a factorable trinomial. Are there other examples? How many?

3. (I.79.3) (Continuation) Find at least three integers that, when put in the blank space, make the expression $x^2 + 4x - \_\_\_$ a factorable trinomial. Are there other examples? How many? What do all these integers have in common?

4. (II.61.6) The line $y = x + 2$ intersects the circle $x^2 + y^2 = 10$ in two points. Call the third quadrant point R and the first-quadrant point E, and find their coordinates. Let D be the point where the line through R and the center of the circle intersects the circle again. Show that triangle RED is a right triangle.

5. (I.68.10) Without using a calculator, solve each of the following quadratic equations:
(a) $(x+4)^2 = 23$ (b) $7x^2 - 22x = 0$ (c) $x^2 - 36x = 205$ (d) $1415 - 16x^2 = 0$

6. (I.76.4) The SSA Ski Club is planning a ski trip for the upcoming long weekend. They have 40 skiers signed up to go, and the ski resort is charging $120 for each person.
(a) Calculate how much money (revenue) the resort expects to take in.
(b) The resort manager offers to reduce the group rate of $120 per person by $2 for each additional registrant, up to a maximum number *m*. For example, if five more skiers were to sign up, all 45 would pay $110 each, producing revenue $4950 for the resort. Fill in the rest of the table at right, and you will discover the manager's value of *m*.
(c) Let x be the number of new registrants. In terms of x, write expressions for the total number of persons going, the cost to each, and the resulting revenue for the resort.
(d) Plot your revenue values versus x, for the relevant values of x. Because this is a *discrete* problem, it does not make sense to connect the dots.
(e) For the resort to take in at least $4900, how many SSA skiers must go on trip?

extras	*persons*	*cost/person*	*revenue*
0			
1			
2			
3			
4			
5	45	110	4950
6			
7			
8			
9			
10			
11			
12			

1. (I.70.5) In the shot-put competition at the SSA-Freeport track meet, the trajectory of Blair's best put is given by the function $h(x) = -0.0186x^2 + x + 5.0,$ where x is the horizontal distance the shot travels, and h is the corresponding height of the shot above the ground, both measured in feet. Graph the function and calculate how far the shot went. What was the greatest height obtained? In the given context, what is the meaning of the "5.0" in the equation?

2. (I.80.5) A small calculator company is doing a study to determine how to price one of its new products. The theory is that the income r from a product is a function of the market price p, and one of the managers has proposed that the quadratic model $r = p \cdot (3000 - 10p)$ provides a realistic approximation to this function.
(a) What was the manager's reasoning in devising this formula?
(b) What is the significance of the value $p = 300$ in this investigation?
(c) Assume that this model is valid and figure out the best price to charge for the calculator. How much income for the company will the sales of this calculator provide?
(d) If the management is going to be satisfied as long as the income from the new calculator is at least \$190000, what range of prices p will be acceptable?

3. (I.75.1) The driver of a red sports car, moving at r feet per second, sees a pedestrian step out into the road. Let d be the number of feet that the car travels, from the moment when the driver *sees* the danger until the car has been brought to a complete stop. The equation $d(r) = 0.75r + 0.03r^2$ models the typical panic-stop relation between stopping distance and speed.
(a) Moving the foot from the accelerator pedal to the brake pedal takes a typical driver three fourths of a second. What does the term $0.75r$ represent in the stopping-distance equation? The term $0.03r^2$ comes from physics; what must it represent?
(b) How much distance is needed to bring a car from 30 miles per hour (which is equivalent to 44 feet per second) to a complete stop?
(c) How much distance is needed to bring a car from 60 miles per hour to a complete stop?
(d) Is it true that doubling the speed of the car doubles the distance needed to stop it?

4. (I.75.2) (Continuation) At the scene of a crash, an officer observed that a car had hit a wall 150 feet after the brakes were applied. The driver insisted that the speed limit of 45 mph had not been broken. What do you think of this evidence?

5. (I.75.5) The equation $y = 50x - 0.5x^2$ describes the trajectory of a toy rocket, in which x is the number of feet the rocket moves horizontally from the launch, and y is the corresponding number of feet from the rocket to the ground. The rocket has a sensor that causes a parachute to be deployed when activated by a laser beam.
(a) If the laser is aimed along the line $y = 20x$, at what altitude will the parachute open?
(b) At what slope could the laser be aimed to make the parachute open at 1050 feet?

1. (I.81.8) Graph the three points (−2, 1), (3, 1), and (0, 7). There is a quadratic function whose graph passes through these three points. Sketch the graph. Find its equation in two ways: First, begin with the equation $y = ax^2 + bx + c$ and use the three points to find the values of *a*, *b*, and *c*. (One of these values is essentially given to you.) Second, begin with the equation $y = a(x-h)^2 + k$ and use the three points to determine *a*, *h*, and *k*. (One of these values is almost given to you.) Your two equations do not look alike, but they should be equivalent. Check that they are.

2. (I.2.4) When describing the growth of a population, the passage of time is sometimes described in generations, a generation being about 30 years. One generation ago, you had two ancestors (your parents). Two generations ago, you had four ancestors (your grandparents). Ninety years ago, you had eight ancestors (your great-grandparents). How many ancestors did you have 300 years ago? 900 years ago? Do your answers make sense?

3. (I.83.8) Graph the equation $y = -2x^2 + 5x + 33.$ For what values of *x*
(a) is $y = 0$? (b) is $y = 21$? (c) is $y \ge 0$?

4. (I.79.4) Combine into one fraction:
(a) $\frac{1}{3}+\frac{1}{7}$ (b) $\frac{1}{15}+\frac{1}{19}$ (c) $\frac{1}{x-2}+\frac{1}{x+2}$
Evaluate your answer to part (c) with $x = 5$ and $x = 17$. How do these answers compare to your answers in parts (a) and (b)?

5. (I.88.1) The perimeter of a rectangular field is 80 meters and its area is 320 square meters. Find the dimensions of the field, correct to the nearest tenth of a meter.

6. (II.48.8) Diagonals *AC* and *BD* of regular pentagon *ABCDE* intersect at *H*. Decide whether or not *AHDE* is a rhombus, and give your reasons.

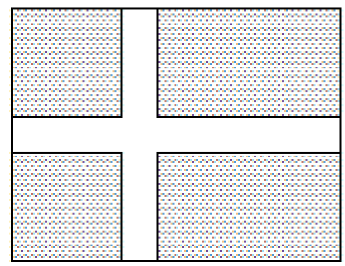
7. (I.77.4) Solve the following quadratic equations:
(a) $x^2 + 6x + 5 = 0$ (b) $x^2 - 7x + 12 = 0$ (c) $x^2 + 3x - 4 = 0$ (d) $x^2 - x - 6 = 0$

8. (I.89.8) We know that the axis of symmetry for a parabola in the form $y = ax^2 + bx + c$ can be found from the formula $x = -\frac{b}{2a}.$ The equation of the axis of symmetry can help us find the *y*-coordinate of the vertex. Make the appropriate substitution, using $x = -\frac{b}{2a},$ and find a formula for the *y*-coordinate of the vertex in terms of *a*, *b*, and *c*.

9. (II.10.5) A 9-by-12 rectangular picture is framed by a border of uniform width. Given that the combined area of picture plus frame is 180 square units, find the width of the border.

1. (I.57.7) Faced with the problem of dividing 5^{24} by 5^8, Brook is having trouble deciding which of these four answers is correct: 5^{16}, 5^3, 1^{16} and 1^3. Your help is needed. Once you have answered Brook's question, experiment with other examples of this type until you are ready to formulate the *common-base principle for division* that tells how to divide b^m by b^n to get another power of b. Then apply this principle to the following situations:
(a) Earth's human population is roughly 7×10^9, and its total land area, excluding the polar caps, is roughly 5×10^7 square miles. If the human population were distributed uniformly over all available land, approximately how many persons would be found per square mile?
(b) At the speed of light, which is 3×10^8 meters per second, how many seconds does it take for the Sun's light to travel the 1.5×10^{11} meters to Earth?

2. (I.90.4) The diagram at right shows the flag of Denmark, which consists of a white cross of uniform width against a solid red background. The flag measures 2 feet 11 inches by 3 feet 9 inches, and the area of the white cross is 5/21 of the area of the whole flag. Use this information to find the width of the white cross.



3. (I.82.1) Solve each of the following by completing the square:
(a) $3x^2 - 6x = 1$ (b) $2x^2 + 8x - 17 = 0$

4. (I.45.1) Wes walks from home to a friend's house to borrow a bicycle, and then rides the bicycle home along the same route. By walking at 4 mph and riding at 8 mph, Wes takes 45 minutes for the whole trip. Find the distance that Wes walked.

5. (I.90.1) Find both solutions to $3x^2 - 7x + 3 = 0$. As a check, you should find that the two answers are reciprocals of one another.

6. (I.61.1) What is the value of $\frac{5^7}{5^7}$? of $\frac{8^3}{8^3}$? of $\frac{c^{12}}{c^{12}}$? What is the value of any number divided by itself? If you apply the *common-base principle for division* dealing with exponents and division, $\frac{5^7}{5^7}$ should equal 5 raised to what power? and $\frac{c^{12}}{c^{12}}$ should equal c raised to what power? It therefore makes sense to define c^0 to be what?

7. (II.11.8) A debt of $450 is to be shared equally among the members of the Outing Club. When five of the members refuse to pay, the other members' shares each go up by $3. How many members does the Outing Club have?

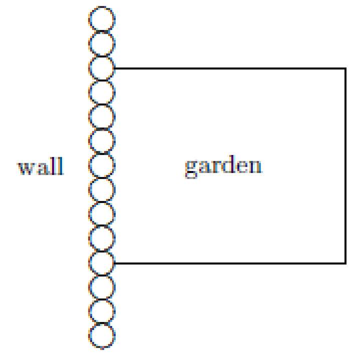
8. (I.86.5) There is a unique parabola whose symmetry axis is parallel to the y-axis, and that passes through the three points (1, 1), (−2, −2), and (0, −4). Write an equation for it. Given any three points, must there be a parabola that will pass through them? Explain.

1. (I.80.7) Combine over a common denominator: (a) $\frac{1}{x-3}+\frac{2}{x}$ (b) $\frac{1}{x-3}+\frac{2}{x+3}$

2. (I.86.9) Sam is a guest on the TV show *Math Jeopardy*, and has just chosen the $300 question in the category "Quadratic Equations." The answer is "The solutions are $x = 3$ and $x = -2$." What question could Sam ask that would win the $300? Is there more than one possible correct question?

3. (II.23.5) Let $A = (0, 12)$ and $B = (25, 12)$. If possible, find coordinates for a point P on the x-axis that makes angle APB a right angle.

4. (I.72.2) Gerry Anium is designing another rectangular garden. It will sit next to a long, straight rock wall, thus leaving only three sides to be fenced. This time, Gerry has bought 150 feet of fencing in one-foot sections. Subdivision into shorter pieces is not possible. The garden is to be rectangular and the fencing (all of which must be used) will go along three of the sides as indicated in the picture.



(a) If each of the two sides attached to the wall were 40 ft long, what would the length of the third side be?
(b) Is it possible for the longest side of the rectangular garden to be 85 feet long? Explain.
(c) Let x be the length of one of the sides attached to the wall. Find the lengths of the other two sides, in terms of x. Is the variable x continuous or *discrete*?
(d) Express the area of the garden as a function of x, and graph this function. For what values of x does this graph have meaning?
(e) Graph the line $y = 2752$. Find the coordinates of the points of intersection with this line and your graph. Explain what the coordinates mean with relation to the garden.
(f) Gerry would like to enclose the largest possible area possible with this fencing. What dimensions for the garden accomplish this? What is the largest possible area?

5. (I.68.6) Add the first two odd positive numbers, the first three odd positive numbers, and the first four odd positive numbers. Do your answers show a pattern? What is the sum of the first n odd positive numbers?

6. (I.68.7) (Continuation) Copy the accompanying tables into your notebook and fill in the missing entries. Notice that the third column lists the differences between successive y-values. Is there a pattern to the column of differences? Do the values in this column describe a linear function? Explain. As a check, create a fourth column that tables the differences of the differences. How does this column help you with your thinking?

x	y
0	0
1	1
2	4
3	9
4	
5	

diff
1
3

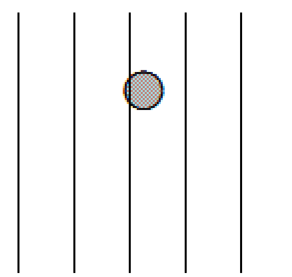
7. (I.68.8) (Continuation) Carry out the same calculations, but replace $y = x^2$ by a quadratic function of your own choosing. Is the new table of differences linear?

1. (I.53.3) Faced with the problem of calculating $\left(5^4\right)^3$, Brook is having trouble deciding which of these three answers is correct: 5^{64}, 5^{12} or 5^7. Once you have answered Brook's question, experiment with other examples of this type until you are ready to formulate the principle that tells how to write $\left(b^m\right)^n$ as a power of *b*.

2. (I.90.6) Graph the nonlinear equation $y = 9 - x^2$, identifying all of the axis intercepts. On the same system of coordinate axes, graph the line $y = 3x - 5$, and identify its axis intercepts. You should see two points where the line intersects the parabola. First estimate their coordinates, then calculate the coordinates exactly by solving the system of simultaneous equations. Which methods of solution work best in this example?

3. (III.37.1) It is well known that multiplication can be *distributed* over addition or subtraction, meaning that $a \cdot (b + c)$ is equivalent to $a \cdot b + a \cdot c$, and that $a \cdot (b - c)$ is equivalent to $a \cdot b - a \cdot c$. It is *not* true that multiplication distributes over multiplication, however, for $a \cdot (b \cdot c)$ is not the same as $a \cdot b \cdot a \cdot c$. Now consider distributive questions about exponents: Is $(b + c)^n$ equivalent to $b^n + c^n$? Explore this question by choosing some numerical examples. Is $(b \cdot c)^n$ equivalent to $b^n \cdot c^n$? Look at more examples.

4. (II.21.1) True or false? $\sqrt{4x} + \sqrt{9x} = \sqrt{13x}$



5. (III.5.6) A coin with a 2-cm diameter is dropped onto a sheet of paper ruled by parallel lines that are 3 cm apart. Which is more likely, that the coin will land on a line, or that it will not?

6. (I.83.10) Sketch the graphs of $y = 3\sqrt{x}$ and $y = x + 2$, and then find their points of intersection. Now solve the equation $3\sqrt{x} = x + 2$ by first squaring both sides of the equation. Do your answers agree with those obtained from the graph?

7. (I.85.1) By averaging 60 miles an hour, Allie made a 240-mile trip in just 4 hours. If Allie's average speed had been only 40 miles per hour, how many hours would the same trip have taken? Record your answer in the given table, then complete the table, knowing that the whole trip was 240 miles.

mph	hrs
10	24
20	12
	8
40	
	4.8
60	4

(a) Multiply 10 by 24, 20 by 12, etc. What do you notice?
(b) Sketch the graph of $y = \dfrac{240}{x}$, where *x* is speed and *y* is time.
(c) What are meaningful values for the speed? Is there a largest one? Is there a smallest?
(d) Is *y* a linear function of *x*? Is *y* a quadratic function of *x*? Explain.

Problem-Based Mathematics II

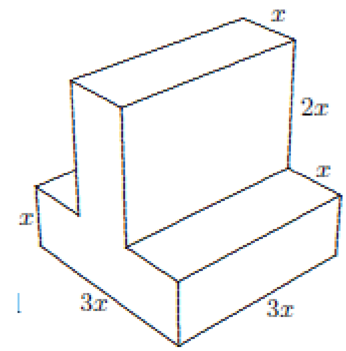
1. (I.62.1) Write the following monomials without using parentheses:

(a) $(ab)^2(ab^2)$ (b) $(-2xy^4)(4x^2y^3)$ (c) $(-w^3x^2)(-3w)$ (d) $(7p^2q^3r)(7pqr^4)^2$

2. Equations involving fractions can be solved by first multiplying both sides of the equation by the least common denominator of all fractions. The transformed equation will then be free of fractions. Try solving the equations below using this method.

(a) $\frac{3x}{5}-\frac{5}{6}=\frac{x}{10}$ LCD = ________ (b) $\frac{5}{x}+\frac{1}{4}=\frac{1}{x}$ LCD = ________

3. (I.48.9) The diagram at the right represents a solid of uniform cross-section. All the lines of the figure meet at right angles. The dimensions are marked in the drawing in terms of *x*. Write simple formulas in terms of *x* for each of the following:
(a) the volume of the solid;
(b) the surface area you would have to cover in order to paint this solid;
(c) the length of decorative cord you would need if you wanted to outline all the edges of this solid.



4. (III.39.10) Explain your opinions of each of the following student responses:
(a) Asked to find an expression equivalent to $x^8 - x^5$, a student responded x^3.
(b) Asked to find an expression equivalent to $\frac{x^8-x^5}{x^2}$, a student responded $x^6 - x^3$.
(c) Another student said that $\frac{x^2}{x^8-x^5}$ is equivalent to $\frac{1}{x^6}-\frac{1}{x^3}$.

5. (I.89.3) Sketch the graphs of $y=\sqrt{x}$ and $y=\sqrt{x-3}$ on the same system of axes. Describe in words how the two graphs are related. Do they intersect?

6. (I.89.4) Sketch the graphs of $y=\sqrt{x}$ and $y=\sqrt{x}-3$ on the same system of axes. (Note: this is *not* the same problem as above.) Describe in words how the two graphs are related. Do they intersect?

7. (I.85.7) Casey loves movies, but has just heard that the Oaks is raising the price of a movie to \$7. Casey decides to buy a DVD player for \$171 and rent movies from Netflix for \$2.50 each instead of going to the Oaks.
(a) In one month during the summer, Casey rents 20 movies. What is the average cost of these movies if the price of the DVD player is included?
(b) Write an equation that expresses *A(n)*, the average cost of a rented movie, as a function of *n*, the number of movies rented.
(c) For what values of *n* is *A(n)* less than the price at the Oaks?
(d) Casey will of course continue to rent movies. Is there a limit to how low the average cost of a rental can go? If so, what is it? If not, explain why not.

1. (I.86.8) Find the solution to each equation:

(a) $\frac{x}{3}+\frac{x}{5}=12$ (b) $\frac{x-2}{-2}=\frac{4x-3}{4}$ (c) $\frac{x+1}{3}+\frac{x-1}{x}=2$

2. Solving equations with fractions sometimes leads to transformed equations whose roots do not work in the original equation. Such roots are called *extraneous roots* and they should be discarded. Solve the equations below and check that each transformed equation contains an extraneous root. Can you offer a reasonable explanation as to why extraneous roots occur?

(a) $\frac{x}{x+3}+\frac{1}{x-3}=1$ (b) $\frac{3}{x^2-7x+10}+2=\frac{x-4}{x-5}$

3. (I.90.5) Alex is making a 4-mile trip. The first two miles were at 30 mph. At what speed must Alex cover the remaining two miles so that the average speed for the entire trip will be:
(a) 50 mph? (b) 55 mph? (c) 59.9mph? (d) 60 mph?

4. (I.91.4) Sketch the graphs of $y=2\sqrt{x}$ and $y = x - 3$, and then find all points of intersection. Now solve the equation $2\sqrt{x}=x-3$ by first squaring both sides of the equation. Do your answers agree with those obtained from the graph?

5. (Continuation) Solve the equations below by using the technique of squaring both sides as in the prior problem. By the way, extraneous roots sometimes occur when solving equations with square roots so be sure to check all of your answers in the original equation.

(a) $\sqrt{2n+3}=n$ (b) $\sqrt{2x^2-7}=5$

6. (III.63.6) How does the graph of $y=f(x)$ compare to the graph of $y+1=f(x-3)$?

7. (III.28.9) Exponents are routinely encountered in scientific work, where they help investigators deal with large numbers:
(a) The human population of Earth is roughly 7000000000, which is usually expressed in *scientific notation* as 7×10^9. The average number of hairs on a human head is 5×10^5. Use scientific notation to estimate the number of human head hairs on Earth.
(b) Light moves very fast—approximately 3×10^8 meters every second. At that rate, how many meters does light travel in one year, which is about 3×10^7 seconds long? This so-called *light year* is used in astronomy as a yardstick for measuring even greater distances.

8. (I.49.3) Most of Conservative Colby’s money was put into a savings account that paid 5% interest a year, but some of it was put into a risky stock fund that paid 16% a year. Colby’s total initial investment in the two accounts was \$10000. At the end of the first year, Colby received a total of \$775 in interest from the two accounts. Find the amount invested in each.

Problem-Based Mathematics II

1. (III.30.12) The diameter of a typical atom is so small that it would take about 10^8 of them, arranged in a line, to reach just one centimeter. It is therefore a plausible estimate that a cubic centimeter could contain about $10^8 \times 10^8 \times 10^8 = \left(10^8\right)^3$ atoms. Write this huge number as a power of 10.

2. (III.34.2) The common-base principle for multiplication predicts that $5^{\frac{1}{2}}$ times $5^{\frac{1}{2}}$ should be 5. Explain this logic, then conclude that $5^{\frac{1}{2}}$ is just another name for a familiar number. Use your calculator to check your prediction. How would you describe the number $6^{\frac{1}{3}}$, given that $6^{\frac{1}{3}} \cdot 6^{\frac{1}{3}} \cdot 6^{\frac{1}{3}}$ equals 6? Formulate a general meaning of expressions like $b^{1/n}$, and use a calculator to test your interpretation on simple examples like $8^{1/3}$ and $32^{1/5}$.

3. (III.35.6) The result of dividing 5^7 by 5^3 is 5^4. What is the result of dividing 5^3 by 5^7, however? By considering such examples, decide what it means to put a *negative* exponent on a base.

4. (III.7.9) A coin of radius 1 cm is tossed onto a plane surface that has been *tessellated* (tiled) by rectangles whose measurements are all 8 cm by 15 cm. What is the probability that the coin lands within one of the rectangles?

5. (III.35.7) Exponents are routinely encountered in science, where they help to deal with small numbers. For example, the diameter of a proton is 0.0000000000003 cm. Explain why it is logical to express this number in scientific notation as 3×10^{-13}. Calculate the surface area and the volume of a proton.

6. (III.38.10) You have deposited $1000 in a money-market account that earns 8 percent annual interest. Assuming no withdrawals or additional deposits are made, calculate how much money will be in the account one year later; two years later; three years later; four years later; *t* years later.

years	money
0	
1	
2	
3	
4	
t	

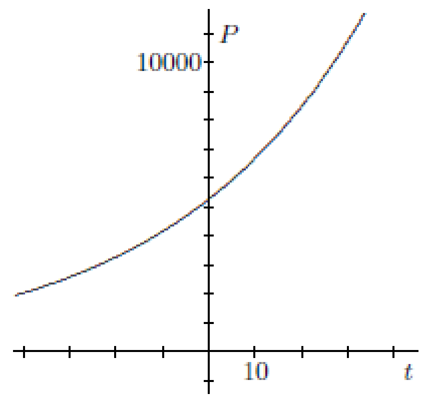
7. (III.40.5) The population of Grand Fenwick has been increasing at the rate of 2.4 percent per year. It has just reached 5280 (a milestone). What will the population be after ten years? after t years? After how many years will the population be 10560?

8. (III.38.1) Invent a division problem whose answer is b^0, and thereby discover the meaning of b^0.

9. (SAT) Let $b \# x$ be defined as b^{2x}. Simplify $(2\#5)\#x$ and $2\#(5\#x)$. Are these expressions the same? Try a few different values for x to check your answer.

10. (Continuation) Simplify the product $(5\#4)\cdot(5\#10)$ without using your calculator. If you write this result as $(5\#a)$ then what is the value of a? Is this surprising?

1. (III.40.6) The figure at right shows part of the graph of $P(t) = 5280(1.024)^t$, a function that describes a small town whose population has been growing at an annual rate of 2.4 percent.
(a) What is $P(0)$, and what is its meaning?
(b) Use the graph to estimate the solution of the equation $P(t) = 10560$.
(c) Calculate $P(-30)$. What does this number mean?
(d) Comment on the part of the graph that lies outside the borders of the illustration. How would it look if you could see it, and what does it mean?



2. (III.40.7) Rewrite each equation so that it has the form "$x = \ldots$" Please do not use "solve."

(a) $x^5 = a^3$ (b) $x^{1/5} = a^3$ (c) $(x+1)^{15.6} = 2.0$ (d) $x^{-2} = a$

3. (III.41.3) Show that $P + Pr + (P + Pr)r = P(1 + r)^2$. Based on your work with exponential growth, interpret the three individual terms on the left side of this equation, and explain why their sum should equal the expression on the right side.

4. (III.41.6) A helium-filled balloon is slowly deflating. During any 24-hour period, it loses 5 percent of the helium it had at the beginning of that period. The balloon held 8000 cc of helium at noon on Monday. How much helium did it contain 3 days later? 4.5 days later? 20 days later? n days later? 12 hours later? k hours later? Approximately how much time is needed for the balloon to lose half its helium? This time is called the *half-life*. Be as accurate as you can.

5. (III.42.4) Convert the following to equivalent forms in which no negative exponents appear:

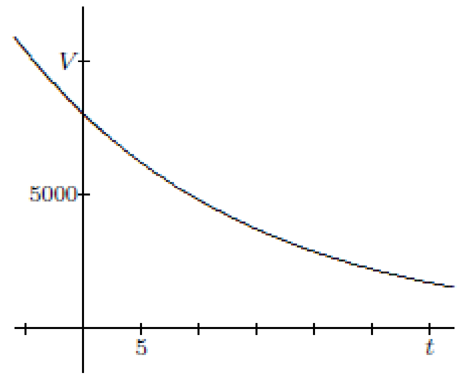
(a) $\left(\frac{2}{5}\right)^{-1}$ (b) $\frac{6}{x^{-2}}$ (c) $\left(\frac{-3}{2}\right)^{-3}$ (d) $\frac{6xy}{3x^{-1}y^{-2}}$ (e) $\left(\frac{2x^2}{3x^{-1}}\right)^{-2}$

6. (III.88.9) The useful function $F(x) = 32 + 1.8x$ converts Celsius temperatures to Fahrenheit temperatures. Confirm this, then find a formula for the function C (which could also be called the *inverse function*, F^{-1}) that converts Fahrenheit temperatures to Celsius temperatures.

7. (II.25.3) Working against a 1-km-per-hour current, some members of the Outing Club paddled 7 km up the Exeter River one Saturday last spring and made camp. The next day, they returned downstream to their starting point, aided by the same one-km-per-hour current. They paddled for a total of 6 hours and 40 minutes during the round trip. Use this information to figure out how much time the group would have needed to make the trip if there had been no current.

8. (III.42.6) In order that a \$10000 investment grow to \$20000 in seven years, what must be the annual rate of interest? Seven years is thus called the *doubling time* for the investment.

1. (III.42.1) The equation at right shows part of the graph of $V(t) = 8000(0.95)^t$. This function tells the story of a shrinking balloon that loses 5 percent of its helium each day.
(a) What is $V(0)$, and what is its significance?
(b) Use the graph to estimate the t-value that solves the equation $V(t) = 4000$.
(c) Calculate $V(-3)$. What does this value mean?
(d) Comment on the part of the graph that lies outside the borders of the illustration. How would it look if you could see it, and what does it mean?



2. (III.43.1) On one system of coordinate axes, graph the equations $y = 3^x$, $y = 2^x$, $y = 1.024^x$, and $y = \left(\frac{1}{2}\right)^x$. What do graphs of the form $y = b^x$ have in common? How do they differ?

3. (III.43.2) Make up a context for the expression $4000(1.005)^{12}$, in which the "12" counts months. In this context, what do the expressions $4000\left((1.005)^{12}\right)^n$ and $4000(1.0617)^n$ mean?

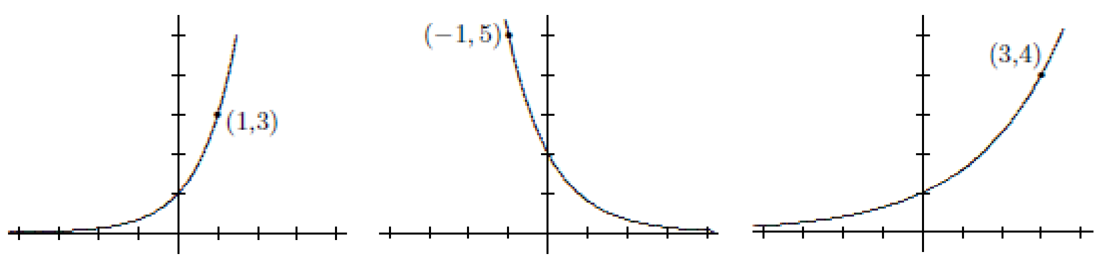
4. (III.45.1) Convert the following to simpler equivalent forms:

(a) $x^6 x^{-6}$ (b) $\left(8a^{-3}b^6\right)^{1/3}$ (c) $\left(\dfrac{x^{1/2}}{y^{2/3}}\right)^6 \left(\dfrac{x^{1/2}}{y^{2/3}}\right)^{-6}$

5. (III.45.9) Explain why calculating $z^{2.5}$ is a square-root problem. What does $z^{0.3}$ mean?

6. (III.46.2) Verify that $(-8)^{\frac{1}{3}}$ can be evaluated, but that $(-8)^{\frac{1}{4}}$ cannot, and explain why $(-8)^{\frac{2}{6}}$ is ambiguous. To avoid difficulties like these, it is customary to restrict the base of an exponential expression to be a positive number when the exponent is not an integer.

7. (III.46.5) Find a plausible equation $y = a \cdot b^x$ for each of the exponential graphs shown below:



8. (III.47.6) Make up a context for the equation $y = 5000(1.005)^x$.

9. (III.47.7) (Continuation) Find the value of x that makes $y = 12500$. Find the value of x that makes $y = 2000$. Interpret these answers in the context you chose.

1. (III.37.7) Find equivalent ways to rewrite (without using a calculator) the following expressions:

(a) $\frac{6a^8}{3a^4}$ (b) $\left(3p^3q^4\right)^2$ (c) $b^{1/2}b^{1/3}b^{1/6}$ (d) $\left(\frac{2x^3}{3y^2}\right)^2$ (e) $\left(d^{1/2}\right)^6$

2. (III.47.9) A constant monthly interest rate of 1.4% is equivalent to what annual interest rate?

3. (III.48.5) Write each of the following numbers as a power of 10. You should not need a calculator.

(a)1000 (b)1000000 (c) 0.01 (d) $\sqrt{10}$ (e) $100\sqrt{10}$ (f) $\frac{1}{\sqrt[3]{100}}$

4. (III.48.6) Using your calculator, write 2017 as a power of 10. Using this answer (and no calculator), write $\frac{1}{2017}$ as a power of 10.

5. (III.48.7) (Continuation) Turn your calculator on, type LOG, (or press the LOG key), type 2017, and press ENTER. Compare the displayed value with the first of your two previous answers.

6. (III.48.8) For each of the following, type LOG followed by the given number, and press ENTER. Interpret the results. By the way, "log" is short for *logarithm*, to be discussed soon.

(a)1000 (b)1000000 (c) 0.01 (d) $\sqrt{10}$ (e) $100\sqrt{10}$ (f) $\frac{1}{\sqrt[3]{100}}$

7. (III.49.2) Using the LOG function of your calculator, solve each of the following for x:

(a) $10^x = 3$ (b) $10^x = 300$ (c) $10^x = 9$ (d) $10^x = 3^{-1}$ (e) $10^x = \sqrt{3}$

You should see a few patterns in your answers—try to explain them.

8. (III.49.7) Rewrite

(a) the *logarithmic equation* 4 = log 10000 as an exponential equation;

(b) the exponential equation $10^{3.30\ldots} = 2016$ as a logarithmic equation.

9. (III.49.8) The function $P(t) = 3960(1.02)^t$ describes the population of Dilcue, North Dakota t years after it was founded.

(a) Find the founding population.

(b) At what annual rate has the population of Dilcue been growing?

(c) Calculate $P(65)/P(64)$.

10. (III.49.9) (Continuation) Try to solve the equation $P(t)$ = 77218. What are you solving for? Can you do it with the tools you have? Explain.

1. (III.50.6) Without using your calculator, solve each of the following equations:
(a) $1000^x = 100000$ (b) $27^x = 243$ (c) $8^x = 32$
Explain why all three equations have the same solution.

2. (III.50.7) Given a positive number *p*, the solution to $10^x = p$ is called the *base-10 logarithm of p*, expressed as $x = \log_{10} p$, or simply $x = \log p$. For example, $10^4 = 10000$ means that 4 is the base-10 logarithm of 10000, or 4 = log 10000. The LOG function on your calculator provides immediate access to such numerical information. Using your calculator for confirmation, and remembering that *logarithms are exponents*, explain why it is predictable that
(a) log 64 is three times log 4;
(b) log 12 is the sum of log 3 and log 4;
(c) log 0.02 and log 50 differ only in sign.

3. (III.51.1) For each description of an exponential function $f(x) = k \cdot b^x$, find *k* and *b*:
(a) $f(0) = 3$ and $f(1) = 12$ (b) $f(0) = 4$ and $f(2) = 1$

4. (III.51.5) *What if the base of an exponential equation isn't 10*? One way of solving an equation like $1.02^x = 3$ is to use your calculator's LOG function to rewrite the equation in the form $\left(10^{0.0086}\right)^x = 10^{0.4771}$. First justify this conversion, then solve $10^{0.0086x} = 10^{0.4771}$.

5. (III.51.6) (Continuation) You have now calculated *the logarithm of 3 using the base 1.02*, for which $x = \log_{1.02} 3$ is the usual notation. The usual ways of reading $x = \log_{1.02} 3$ are "log base 1.02 of 3" or "log 3, base 1.02", or "the base-1.02 logarithm of 3", or "log to the base 1.02 of 3." Because your calculator does not have a button devoted to base-1.02 logarithms, the desired value was obtained as a quotient of two base-10 logarithms. Explain. By the way, do you recall a context for the equation $1.02^x = 3$?

6. (III.49.1) Explain your opinions of each of the following student responses:
(a) Asked for an expression equivalent to $x^3 + x^{-3}$, a student responded x^0.
(b) Asked for an expression equivalent to $\left(x^{-1} + y^{-1}\right)^{-2}$, a student responded $x^2 + y^2$.

7. (III.52.4) For the first 31 days of your new job, your boss offers you two salary options. The first option pays you \$1000 on the first day, \$2000 on the second day, \$3000 on the third day, and so on—in other words, \$1000*n* on the *nth* day. The second option pays you one penny on the first day, two pennies on the second day, four pennies on the third day—the amount doubling from one day to the next. Which option do you prefer, and why?

8. (III.52.5) (Continuation) You have chosen the second payment option, and—on the thirty-first day—your boss pays you the wages for that day in pennies. You wonder whether all these coins are going to fit into your dormitory room, which measures 12 feet by 15 feet by 8 feet. Verify that a penny is 0.75 inch in diameter, and that seventeen of them make a stack that is one inch tall. Use this information to decide whether the pennies will all fit.

Problem-Based Mathematics II

1. (III.61.5) Without using a calculator, simplify:
(a) $\log\left(10^{-4}\right)$ (b) $10^{\log 10000}$ (c) $\log\left(10^{-2.48}\right)$ (d) $10^{\log 4.8}$

2. (III.52.6) Solve $2^x = 1000.$ In other words, find $\log_2 1000,$ the base-2 logarithm of 1000.

3. (III.52.7) You now know how to calculate logarithms by using 10 as a *common base*. Use this method to evaluate the following. Notice those for which a calculator is not necessary.
(a) $\log_8 5$ (b) $\log_5 8$ (c) $\log_5 \sqrt{5}$ (d) $\log_{1.005} 2.5$ (e) $\log_3\left(\frac{1}{9}\right)$

4. (III.52.9) There is an exponential function $f(t) = k \cdot b^t,$ for which the graph $y = f(t)$ contains the points (1, 6) and (3, 24). Find k and b.

5. (III.52.10) You have come to associate a function such as $p(t) = 450 \cdot (1.08)^t$ with the size of something that is growing (exponentially) at a fixed rate. Could such an interpretation be made for the function $d(t) = 450 \cdot 2^t$? Explain.

6. (III.53.2) Given $10^{0.301} \approx 2$ and $10^{0.477} \approx 3,$ solve without a calculator:
(a) $10^x = 6;$ (b) $10^x = 8;$ (c) $10^x = \frac{2}{3};$ (d) $10^x = 1.$

7. (III.53.3) Given that $0.301 \approx \log 2$ and that $0.477 \approx \log 3$, you should not need a calculator to evaluate
(a) $\log 6$; (b) $\log 8$; (c) $\log(2/3)$; (d) $\log 1$.

8. (III.53.5) Given that $m = \log a$ and $n = \log b$,
(a) express a as a power of 10, and express b as a power of 10;
(b) use your knowledge of exponents to express ab as a power of 10;
(c) conclude that $\log(ab) = \log a + \log b$.

9. (III.53.6) (Continuation) Justify the rules: (a) $\log\left(a^r\right) = r\log a$ (b) $\log\left(\frac{a}{b}\right) = \log a - \log b$

10. (III.82.5) An *arithmetic sequence* is a list in which each term is obtained by adding a constant amount to its predecessor. For example, the list 4.0 , 5.2 , 6.4 , 7.6 , . . . is arithmetic. The first term is 4.0; what is the fiftieth? What is the millionth term? What is the n^{th} term?

11. (III.82.6) Suppose that $a_1, a_2, a_3, \ldots$ is an arithmetic sequence, in which $a_3 = 19$ and $a_{14} = 96$. Find a_1.

12. (III.93.4) Explain how $5^{1/256}$ can be calculated using only the square root key on your calculator.

1. (III.53.7) The function $F(x) = 31416(1.24)^x$ describes the number of mold spores found growing on a pumpkin pie *x* days after the mold was discovered.
(a) How many spores were on the pie when the mold was first discovered?
(b) How many spores were on the pie two days before the mold was discovered?
(c) What is the daily rate of growth of this population?
(d) What is the hourly rate of growth?
(e) Describe the spore count on the same pie by the function $G(x)$, where *x* counts the number of hours since the mold was discovered on the pie.

2. (III.54.5) Another approach to solving an equation like $5^x = 20$ is to *calculate base-10 logarithms of both sides of the equation*. Justify the equation $x \log 5 = \log 20$, then obtain the desired answer in the form $x = \frac{\log 20}{\log 5}$. Evaluate this expression. Notice that $\log_5 20 = \frac{\log 20}{\log 5}$.

3. (III.54.6) Write an expression for $\log_a N$ that refers only to base-10 logarithms, and explain.

4. (III.54.7) Asked to simplify $\frac{\log 20}{\log 5}$, Brett replied "log 4." What do you think of this answer?

5. (III.87.12) Given the function $P(x) = 3960(1.06)^x$, find a formula for the *inverse function* $P^{-1}(x)$. In particular, calculate $P^{-1}(5280)$, and invent a context for this question. Graph $y = P^{-1}(x)$.

6. (III.56.8) Ryan spills some soda and neglects to clean it up. When leaving for spring break, Ryan notices some ants on the sticky mess but ignores them. Upon returning seventeen days later, Ryan counts 3960 ants in the same place. The next day there are 5280 ants. Assuming that the size of the ant population can be described by a function of the form $F(t) = a \cdot b^t$, calculate the number of ants that Ryan saw when leaving for spring break.

7. (III.60.1) What is half of 2^{40}? What is one third of 3^{18}?

8. (III.57.3) A *geometric sequence* is a list in which each term is obtained by multiplying its predecessor by a constant. For example, 81, 54, 36, 24, 16, . . . is geometric, with constant multiplier 2/3. The first term of this sequence is 81; what is the 40^{th} term? the millionth term? the n^{th} term? Check your formula for $n = 1$, $n = 2$, and $n = 3$.

9. (III.78.3) Write down the first few terms of any geometric sequence of positive terms. Make a new list by writing down the logarithms of these terms. This new list is an example of what is called an *arithmetic sequence*. What special property does it have?

Problem-Based Mathematics II

1. (III.57.5) Given that $\log_c 8 = 2.27$ and $\log_c 5 = 1.76$, a calculator is *not needed* to evaluate

(a) $\log_c 40$ (b) $\log_c \left(5/8\right)$ (c) $\log_c 2$ (d) $\log_c \left(5^m\right)$ (e) $\log_c 0.04$

2. (III.58.3) Without calculator, find *x*: (a) $\log_4 x = -1.5$ (b) $\log_x 8 = 16$ (c) $27 = 8\left(x-2\right)^3$

3. (III.58.8) When $10^{3.5623}$ is evaluated, how many digits are found to the left of the decimal point? You should be able to answer this question without using your calculator.

4. (III.59.8) Calculate (a) $\log_5 2016$ (b) $\log_{1.005} 3$ (c) $\log_{0.125} 64$ (d) $\dfrac{\log 2016}{\log 5}$

5. (III.60.2) Let $R(t) = 55\left(1.02\right)^t$ describe the size of the rabbit population in the SSA woods *t* days after the first of June. Use your calculator to make a graph of this function inside the window $-50 \le t \le 100$, $0 \le R(t) \le 500$. (You will need to work with the variables *x* and *y* instead of *t* and *R*, of course.) What is the *y*-intercept of the graph, and what does it signify? Does your calculator show an *x*-intercept? Would it show an *x*-intercept if the window were enlarged?

6. (III.60.3) (Continuation) Choose a point on the graph that is very close to the *y*-intercept, then find the slope between this point and the *y*-intercept. This will give you an estimate for the rate (in rabbits per day) at which the population is growing on June 1st. In the same way, estimate the rate at which the population is growing on September 1st, which is 92 days after June 1st. Explain how your two answers are both consistent with the given 2% growth rate.

7. (III.61.3) Fill in the missing entries in the two tables shown at right. Do this without a calculator.

x	10^x
-3	
	1
0.5	3.162
	100
3	
	1996

x	$\log x$
0.001	
	0
3.162	
	2
1000	
1996	3.300

8. (III.61.4) (Continuation) What do your results tell you about the graphs of $y = 10^x$ and $y = \log x$?

9. (III.61.11) Compare the graph of $y = \log x$ and the graph of $y = \log\left(10x\right)$. How are they related?

10. (III.62.5) Draw the graph of the equation $y = \log_2 x$. How does this graph compare to the graph of the equation $y = 2^x$?

11. (III.56.7) What happens when you ask your calculator to evaluate log(−7)? Explain.

12. (III.66.6) Let $f(x) = 5^x$. Calculate $\dfrac{f(1.003) - f(1.000)}{0.003}$ and then explain its significance.

1. (III.64.3) Simplify: (a) $\left(3^{-1}+4^{-1}\right)^{-1}$ (b) $\frac{6^{4000}}{12^{2000}}$ (c) $7^u 7^u$ (d) $\sqrt{64x^{16}}$ (e) $2^m 3^{-m}$

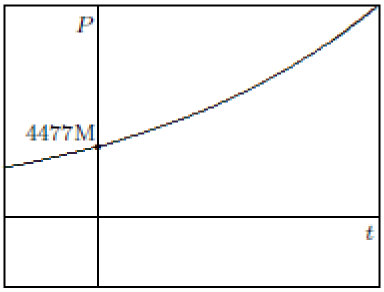
2. (III.64.1) What is the *x*-intercept of the graph of $y=\log(x-3)-1$? How does the graph of $y=\log(x-3)-1$ compare with the graph of $y=\log x$?

3. (III.64.2) Write 10^{2016} as a power of 2.

4. (III.64.4) Explain how to use your calculator's base-10 log function to obtain base-16 logarithms.

5. (III.64.8) A function of the form $H(x)=a\cdot b^x$ has the property that $H(1) = 112$ and $H(3) = 63$. Find the values $H(0)$ and $H(4)$.

6. (III.66.7) How does the graph of $y=\left(\frac{1}{2}\right)^x$ compare with the graph of $y=2^x$? What features do these curves have in common? How are the slopes of these curves related at their common *y*-intercept?



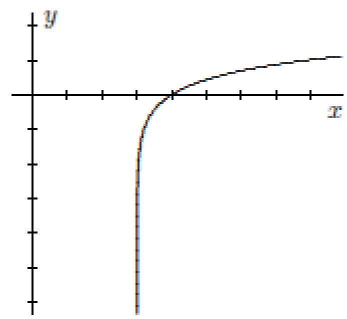
7. (III.72.1) The size of the Earth's human population at the beginning of the year 1980 + *t* is described by the function $P(t)=4477000000(1.0176)^t$, whose graph is shown at right. At what rate is $P(t)$ increasing when $t = 10$? There are at least two ways to interpret this question; give an answer for each of your interpretations.

8. (III.72.2) (Continuation) At what rate is $P(t)$ increasing when $t = 20$? There are at least two ways to interpret this question, so give an answer for each of your interpretations. Does it make sense to say that the Earth's human population is growing at a *constant rate*? Discuss.

9. (III.64.9) Given that $\log_4 x$ is somewhere between −1.0 and 0.5, what can be said about *x*?

10. (III.56.6) Use exponential notation to rewrite the following:
(a) $x^2\sqrt{x}$ (b) $\frac{x}{\sqrt{x}}$ (c) $\frac{\sqrt{x}}{\sqrt[3]{x}}$ (d) $\sqrt[3]{x^3y^4z^5}$

11. (III.65.6) The point (3, 8) is on the graph of $y=2^x$. What is the corresponding point on the graph of the inverse function $y=\log_2 x$? Find four more pairs of points like these.

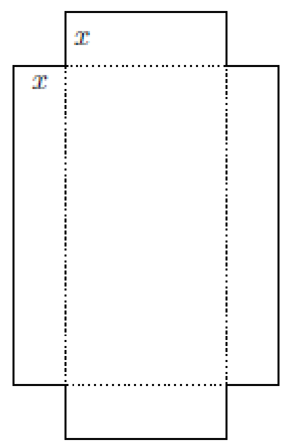


1. (III.62.9) The equation graphed at right is $y = \log_5(x-3)$. What is the *x*-intercept of this graph? There are many vertical lines that do not intersect this graph; which one of them is farthest to the right? For what *x*-values does the equation make sense? What *x*-value corresponds to $y = 1$? to $y = 2$? to $y = 3$?

2. (III.62.10) (Continuation) How does the given graph compare to the graph of $y = \log_5 x$? How does the given graph compare to the graph of $y = \log_5(x+2)$?

3. (III.73.1) Explain why the expression $\log(a) - \log(b)$ should not be confused with $\log(a-b)$. Rewrite $\log(a) - \log(b)$ in an equivalent logarithmic form.

4. (III.73.6) What else can be said about the positive number *p*, given that
(a) $0.0 < \log_7 p < 1.0$? (b) $0.0 < \log_{1/2} p < 1.0$? (c) $0.0 < \log_b p < 1.0$?

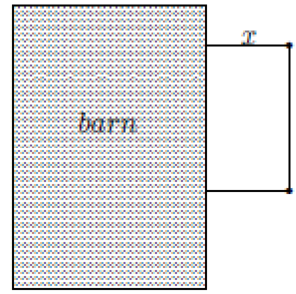


5. (III.90.3) A rectangular box (with no top face) can be formed by cutting four congruent squares from the corners of a rectangular sheet of cardboard and then folding up the sides, as shown in the figure at right. The volume of the resulting box depends on the size of the cutouts. For example, suppose that the original sheet of cardboard is 5 cm by 8 cm, and that the four cutouts are *x* cm square.
(a) Find a formula for the volume $V(x)$ of the resulting box.
(b)What is the *domain* of this function? What does its graph look like?
(c) Find the two solutions to the equation $V(x) = 8$.
(d) Find the maximum value of $V(x)$.
(e) Show that the equation $V(x) = 18$ has only one solution

6. (III.63.4) Write an equation for the curve obtained by shifting the curve $y = 2^x$
(a) three units to the right; (b) five units down.
For each, identify *x*- and *y*-intercepts and other significant features.

7. (III.86.3) Let $f(x) = \sqrt{9-x^2}$ for $-3 \le x \le 3$. Sketch the graph $y = f(x)$, then compare that graph with the graphs of the following related functions. Be prepared to discuss the role of the parameter 2 in each; in particular, how does it affect the domain and the range?
(a) $y = 2f(x)$ (b) $y = f(2x)$ (c) $y = f(x) + 2$ (d) $y = f(x+2)$

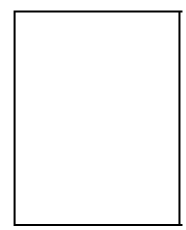
8. (III.63.5) Find the *y*-intercept of the graph of $y + 1 = 2^{x-3}$. How does the graph of $y + 1 = 2^{x-3}$ compare with the graph of $y = 2^x$? How about the graph of $y = 2^{x-3} - 1$?



1. (III.91.11) A farmer wants to enclose the largest possible rectangular area, using 180 feet of chicken-wire, two fence posts, and one side of a long barn. The top view at right shows that the area of the pen depends on the dimension *x*. What is the domain of possible *x*-values? What choice for *x* maximizes the area?

2. (III.91.5) Compare the domains and ranges of the functions $f(x) = 2\log x$ and $g(x) = \log\left(x^2\right)$.

3. (III.88.7) Soda cans usually hold 355 cc (12 fluid ounces, that is) of liquid. It seems likely that the manufacturers of these cans choose the dimensions so that the material required to enclose those 355 cc is as small as possible. Let us find out.
(a) Give an example of a right circular cylinder, choosing its radius and height to make the volume 355, then calculate the total surface area of your cylinder, in square cm.
(b) Express the height of such a cylinder as a function of its radius *r*.
(c) Find the value of *r* that gives a cylinder of volume 355 the smallest total surface area that it can have. Calculate the resulting height, and compare these two dimensions with the actual dimensions of a soda can.



4. (III.66.4) A rectangular sheet of paper (such as the one shown in the figure at right) has thickness 0.003 inches. Suppose that it is folded in half, then folded in half again, then folded in half again—fifty times in all. How thick is the resulting wad of paper? How many folds would it take to reach the Sun?

5. (III.95.10) Simplify $\left(\log_a b\right)\left(\log_b a\right)$.